UNIVERSIDAD CATÓLICA DEL NORTE
FACULTAD DE CIENCIAS
DEPARTAMENTO DE MATEMÁTICAS

SOBRE TRANSITIVIDAD ROBUSTA EN VARIEDADES CON BORDE

Memoria para optar al grado de Doctor en Ciencias Mención Matemática

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"Un matemático es una máquina

para transformar café en teoremas"

<u>Paul Erdos.</u>

"En el arte, nada que merezca la pena se puede hacer sin genio;

en ciencia, incluso una capacidad muy modesta

puede contribuir a un logro supremo"

<u>Bertrand Russell.</u>

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Resumen

Según lo establecido por Morales, Pacifico y Pujals [53], todo conjunto singular C^1-robusto transitivo de un flujo en una 3-variedad compacta sin borde, tiene una estructura hiperbólica-singular,es decir, es parcialmente hiperbólico y expande volumen en la dirección central; es un atractor para el campo X o bien para el campo $-X$; sus singularidades son de tipo-Lorenz. Dos preguntas naturales surgen a partir de este resultado: ¿Es válido si cambiamos C^1-robusto transitivo, por C^k-robusto transitivo, $k > 1$?. ¿Son válidos estos resultados en el contexto de variedades con borde?.

En este trabajo consideramos campos de vectores en variedades con borde. Asumiendo suficiente diferenciabilidad, mostramos la existencia de conjuntos robustos transitivos que no satisfacen las propiedades descritas anteriormente. La existencia de tales conjuntos está relacionada a la existencia de ciclos singulares cuyos elementos críticos pertenecen al borde de la variedad. También mostramos la existencia de conjuntos atractores robustos los cuales no tienen una estructura hiperbólica-singular y sus singularidades no son de tipo-Lorenz. Los resultados obtenidos son los siguientes:

(a) Si $X \in \mathscr{X}^\infty(M, \partial M)$ exhibe un ciclo singular genérico asociado a una singularidad $\sigma \in \partial M$ con autovalores reales λ_{ss}, λ_s y λ_u satisfaciendo

$$\lambda_{ss} < \lambda_s < 0 < \lambda_u \quad \text{y} \quad \lambda_s - \lambda_{ss} - 2 \cdot \lambda_u > 0,$$

entonces para $k \geq 2$ suficientemente grande (cuya grandeza depende sólo de los autovalores de σ) X exhibe un conjunto C^k-robusto transitivo conteniendo a σ.

En particular, para toda 3-variedad compacta conexa con borde M existe $X \in \mathscr{X}^\infty(M, \partial M)$ exhibiendo un conjunto C^k-robusto transitivo el cual no es hiperbólico-singular para X ni para $-X$. Además tal conjunto no es atractor ni repulsor y tiene una singularidad que no es tipo-Lorenz.

(b) Para toda 3-variedad compacta conexa con borde M existe $X \in \mathscr{X}^\infty(M, \partial M)$ tal que para r suficientemente grande, X presenta un conjunto atractor C^r-robusto que no es hiperbólico-singular. Además tal conjunto tiene singularidades que no son tipo-Lorenz.

(c) Existe un campo de vectores C^∞ definido sobre una variedad compacta tri-dimensional, que exhibe un conjunto atractivo hiperbólico-singular conteniendo una singularidad que no es tipo-Lorenz y un conjunto repulsor hiperbólico no-trivial.

Estos resultados (items (a) y (b)) muestran que las propiedades obtenidas por Morales, Pacifico, Pujals [53], [50], referentes a la relación existente entre robustez transitiva, atractores robustos e hiperbolicidad-singular no son válidos en el contexto de 3-variedades compactas con borde y alta diferenciabilidad. En trabajos aún en desarrollo abordamos el caso de C^1-diferenciabilidad.

Índice general

Introducción

Durante mucho tiempo el objetivo en la teoría de los sistemas dinámicos ha sido describir y caracterizar sistemas que exhiban propiedades preservables bajo pequeñas perturbaciones. Una piedra angular en esta dirección fue la Conjetura de la Estabilidad Estructural (ver Palis-Smale [61]), que establece que aquellos sistemas que a menos de un cambio de coordenadas continuo en el espacio de fase son idénticos a todos los sistemas cercanos, son los hiperbólicos. Condiciones suficientes para la estabilidad estructural fueron provadas por Robbin [67] (para $r \geq 2$), de Melo [43] y Robinson [69] (para $r = 1$). Su condición necesaria fue reducida a mostrar que estabilidad estructural implica hiperbolicidad (Robinson [68]); que fue probada por Mañé [38] en el caso discreto (para $r = 1$) y por Hayashi [28] en el contexto de flujos (para $r = 1$).

Esto tiene importantes consecuencias debido a la existencia de una rica teoría para sistemas hiperbólicos, describiendo sus propiedades geométricas y ergódicas. En particular, para sistemas estructuralmente estables, se tiene por el teorema de la descomposición espectral de Smale [76], una descripción del conjunto no errante como un número finito de conjuntos maximales invariantes compactos disjuntos y transitivos, y donde cada una de estas piezas es bien entendida desde el punto de vista determinístico y estadístico. Más aún, tal descomposición persiste bajo perturbaciones C^1. Esto naturalmente conduce al estudio de conjuntos transitivos aislados que permanecen transitivos para todos los sistemas cercanos (Robustez), propiedad que también da información dinámica de ellos. ¿Existen conjuntos robustamente transitivos no hiperbolicos? ¿Qué se puede decir sobre la dinámica de conjuntos robustamente transitivos?, ¿Existe una caracterización de tales conjuntos?.

En el caso de flujos en dimensión tres, un sorprendente ejemplo es el atractor de Lorenz

[36], dado por la solución del siguiente campo polinomial de vectores en $\mathbb{R}^3$:

$$X(x,y,z) = \begin{cases} \dot{x} &= -\alpha x + \alpha y \\ \dot{y} &= \beta x - y - xz \\ \dot{z} &= -\gamma z + xy, \end{cases} \tag{1}$$

donde α, β, γ son parámetros reales. Experimentos numéricos realizados por Lorenz (para $\alpha = 10$, $\beta = 28$ y $\gamma = \frac{8}{3}$) sugieren la existencia, en un sentido robusto, de un atractor hacia el cual tiende toda trayectoria positiva del sistema anterior. Lo más relevante es el hecho que este atractor contiene al punto de equilibrio $(0,0,0)$ y órbitas periódicas acumulándose en él, y que por lo tanto no puede ser hiperbólico. Cabe destacar que, sólo ahora, tres décadas y media después de este notable trabajo, Tucker [82] probó que la solución de (1) satisface tal propiedad para valores cercanos a los considerados por Lorenz.

Por otro lado, a mediados de los setenta, en un intento por comprender a partir de un enfoque geométrico la dinámica presentada por Lorenz, es mostrada la existencia de atractores robustos no hiperbólicos para flujos (introducidos en [2], [24], [86]), los cuales ahora son llamados modelos geométricos para los atractores de Lorenz. En particular, ellos exhiben de manera robusta, un conjunto atractivo transitivo con una singularidad. Para tal singularidad los autovalores λ_u,λ_s y λ_{ss} son reales y satisfacen $\lambda_{ss} < \lambda_s < 0 < -\lambda_s < \lambda_u$.

La comprensión del fenómeno de la coexistencia robusta de singularidades y órbitas periódicas regulares de manera transitiva, ha tenido un gran progreso con los trabajos de Morales, Pacifico y Pujals, [53] y [50] donde se muestra que conjuntos singulares robustos para flujos en dimensión tres son atractores o repulsores hiperbólicos-singulares y que de hecho ellos comparten todas las propiedades geométricas de los atractores de Lorenz geométricos.

Para formalizar mas precisamente los resultados, necesitaremos introducir definiciones básicas y la notación correspondiente.

Sea M una 3-variedad compacta conexa y sea ∂M su borde. Denotemos por $\mathscr{X}^k(M, \partial M)$, $k \geq 1$, el espacio de los campo de vectores de clase C^k en M tangentes a ∂M y el cual está equipado con la topología C^k estándar. En caso que ∂M es vacío, este espacio es denotado por $\mathscr{X}^k(M)$. Fijamos $X \in \mathscr{X}^k(M, \partial M)$ y denotamos por X_t, $t \in \mathbb{R}$, el flujo generado por X en M.

La *órbita* de X por $p \in M$ es el conjunto $O_X(p) = \{X_t(p) : t \in \mathbb{R}\}$. El ω-*límite* de un punto $p \in M$, es el conjunto de los puntos $q \in M$ tales que existe una sucesión $t_n \to \infty$ con $X_{t_n}(p) \to q$. Una *singularidad* de X es un punto $\sigma \in M$ tal que $X(\sigma) = \{0\}$ (equivalentemente, $O_X(\sigma) = \{\sigma\}$). La órbita $O = O_X(p)$ es una *órbita periódica* de X si $X_T(p) = p$, para algún $T > 0$ minimal. Una *órbita cerrada* (*elemento crítico*) de X es una singularidad o una órbita periódica de X.

Un conjunto compacto $\Lambda \subset M$ es *invariante* por X si para todo $t \in \mathbb{R}$, $X_t(\Lambda) = \Lambda$; *no-trivial* si no es una órbita cerrada ; *transitivo* si es el conjunto ω-límite de uno de sus puntos; y *aislado* si existe una vecindad compacta U de Λ (llamada *bloque aislante*) tal que

$$\Lambda = \bigcap_{t \in \mathbb{R}} X_t(U).$$

Un conjunto aislado Λ es C^k-*robusto transitivo* si exhibe un bloque aislante U tal que para todo $Y \in \mathscr{X}^k(M, \partial M)$ C^k-cercano a X, la continuación de Λ

$$\Lambda_Y = \bigcap_{t \in \mathbb{R}} Y_t(U)$$

es un conjunto transitivo no-trivial de Y.

Un conjunto Λ es *atractivo* si es aislado y tiene un bloque aislante U positivamente invariante, es decir, para todo $t \geq 0$,

$$X_t(U) \subset U.$$

Un conjunto Λ es un *atractor* si es atractivo y transitivo.

La definición anterior de conjunto atractivo no es estándar. Cabe mencionar que muchos autores llaman atractor al que estamos llamando conjunto atractivo. Ver Milnor [46] donde varias nociones de atractores son discutidas.

Un Conjunto compacto Λ de X es *hyperbólico* si su fibrado tangente se descompone en una suma directa invariante $E^s_\Lambda \oplus E^X_\Lambda \oplus E^u_\Lambda$ donde E^X_Λ es el subfibrado generado por X sobre Λ y E^s_Λ (resp. E^u_Λ) es un subfibrado contractivo (resp. expansivo) para la derivada de X_t cuando $t \to \infty$, esto es $||DX_t(x)|_{E^s_x}|| \leq C\lambda^t$ (resp. $||DX_{-t}(x)|_{E^u_x}|| \leq C\lambda^t$). Entonces, la Teoría de Variedades Invariantes [31] afirma que a través de cualquier punto $x \in \Lambda$ pasa una variedad estable fuerte (resp. inestable) $W^{ss}(x)$ (resp. $W^{uu}(x)$) tangente en x al subespacio E^s_x (resp. E^u_x). Por saturación de éstas variedades obtenemos la variedad estable e inestable $W^s(x)$ y $W^u(x)$ de x respectivamente. Una órbita cerrada es hiperbólica si y sólo si lo es como un conjunto compacto invariante.

Un conjunto compacto invariante Λ de X es *parcialmente hiperbólico* si su fibrado tangente se descompone en una suma directa invariante $F_\Lambda \oplus F^c_\Lambda$ donde F_Λ es contractivo o expansivo y domina a F^c_Λ, esto es la menor contracción en F_Λ es mayor que la mayor contracción en F^c_Λ o la menor expansión en F_Λ es mayor que la mayor expansión en F^c_Λ respectivamente. Decimos que el subfibrado central F^c_Λ *expande volumen* cada vez que la derivada del flujo expande exponencialmente el volumen a lo largo de F^c_Λ cuando $t \to \infty$. Un conjunto *hiperbólico-singular* es un conjunto parcialmente hiperbólico cuyas singularidades son hiperbólicas y cuyo sub-fibrado central expande volumen. Una singularidad $\sigma \in M$ de X es *tipo-Lorenz* si los autovalores de $DX(\sigma)$ son reales y satisfacen $\lambda_{ss} < \lambda_s < 0 < \lambda_u$ y $\lambda_s + \lambda_u > 0$.

Dos preguntas naturales surgen a partir de los trabajos de Morales, Pacifico y Pujals: ¿ Son válidos si cambiamos C^1-robusto transitivo, por C^k-robusto transitivo, $k > 1$?. Para flujos en 3-variedades compactas con borde, ¿todo conjunto C^k-robusto transitivo conteniendo singularidades es atractor o repulsor hiperbólico-singular y todas sus singularidades son tipo-Lorenz?. En este trabajo nos centramos en la segunda interrogante.

Otros resultados obtenidos por Morales, Pacifico y Pujals [50] para campos de vectores de clase C^1 en una 3-variedad compacta cerrada que apuntan en esta dirección son los siguientes:

(a) Todo conjunto C^1-robusto transitivo conteniendo singularidades de X es atractor o repulsor propio.

(b) Toda singularidad de un atractor C^1-robusto de X es tipo-Lorenz.

(c) Todos los atractores C^1-robustos de X conteniendo singularidades son conjuntos hiperbólicos-singulares.

(d) Sea Λ un conjunto aislado de un campo de vectores X de clase C^r, $r \geq 1$. Decimos que Λ *contiene rubustamente la variedad inestable de un elemento crítico* si existen: x_0 elemento crítico hiperbólico de X en Λ, un bloque aislante U de Λ y una vecindad $\mathscr{U}$ de X en el espacio de los campo de vectores de clase C^r tal que para todo $Y \in \mathscr{U}$,

$$W^u_Y(x_0(Y)) \subset U.$$

Si Λ es un conjunto transitivo aislado de X tal que todos sus elementos críticos son hiperbólicos y contiene robustamente la variedad inestable de un elemento crítico entonces Λ es un atractor.

(e) Si se satisface que:

(i) Λ es C^1-robusto transitivo

(ii) Todo elemento crítico en Λ es hiperbólico y

(iii) Λ tiene una singularidad cuya variedad inestable es unidimensional,

entonces Λ es un atractor.

En este trabajo mostramos que en el contexto de variedades con borde la situación es diferente. Particularmente como consecuencia del teorema principal, obtenemos que los incisos (a), (d) y (e) no son válidos (para alta diferencialbilidad). Además, exhibimos un ejemplo simple que muestra que (b) y (c) no son válidos para alta diferenciabilidad.

El presente trabajo se concentra en el estudio de la dinámica en una vecindad de un ciclo singular. Por *ciclo singular* de X nos referimos a un conjunto Γ consistiendo de singularidades y órbitas periódicas, digamos $\{\sigma_0, \sigma_1, \cdots, \sigma_n\}$ conectadas de manera cíclica ($\sigma_0 = \sigma_n$) por órbitas regulares $\gamma_i' s$ de modo que $\gamma_i \subset W^u(\sigma_i) \cap W^s(\sigma_{i+1})$. Un ciclo es singular hiperbólico si los elementos críticos son hiperbólicos. En caso que $\sigma_0 = \sigma$ sea la única singularidad, decimos que el ciclo singular Γ está *asociado* a σ. Estamos interesados en ciclos singulares asociados a singularidades con autovalores reales $\lambda_u, \lambda_s, \lambda_{ss}$ satisfaciendo las relaciones de autovalores

$$\lambda_{ss} < \lambda_s < 0 < \lambda_u, \text{ and } \lambda_s - \lambda_{ss} - 2\lambda_u > 0. \tag{2}$$

Esta última condición en los autovalores nos permitirá construir en torno al ciclo un sistema de coordenadas C^3 de tal manera que la dinámica asociada será dada por una aplicación con derivada Schwarziana negativa.

La singularidad σ está equipada con varias variedades invariantes. Una variedad estable 2-dimensional $W^s(\sigma)$ y una variedad inestable 1-dimensional $W^u(\sigma)$. Existe también la variedad central-inestable $W^{cu}(\sigma)$ tangente en σ al subespacio asociado a $\{\lambda_s, \lambda_u\}$. La Variedad central-inestable no es única pero todas ellas contienen a $W^u(\sigma)$. Mas aún, el sub-fibrado tangente TW^{cu} está bien definido a lo largo de $W^u(\sigma)$. Decimos que el ciclo singular es *simple* si $W^{cu}(\sigma)$ es transversal a $W^s(\sigma_1)$ a lo largo de γ_0 y para todo $i > 0$, $W^u(\sigma_i)$ es transversal a $W^s(\sigma_{i+1})$ a lo largo de γ_i. Estas propiedades son *genéricas*. Note que estas condiciones implican que $\sigma_1 = O$ es una órbita periódica.

La dinámica en una vecindad del ciclo puede ser entendida a través de las transformaciones de Poincaré asociadas. Las hipótesis garantizan la existencia de una foliación invariante $\mathscr{F}$ ($\mathscr{F}$ es *invariante* si la imagen toda hoja L de $\mathscr{F}$ está contenida en alguna hoja $\tilde{L}$ de $\mathscr{F}$), llamada foliación estable fuerte, que permite reducir dichas transformaciones de Poincaré a aplicaciones del intervalo. En general tal foliación es sólo continua, pero asumiendo X suficientemente diferenciable, esta es diferenciable. Denotemos por $\mathscr{A}$ al conjunto de aplicaciones del intervalo asociadas al estudio de los ciclos de acuerdo a lo anteriormente establecido. En el caso en que la singularidad sea contractiva ($\lambda_s + \lambda_u < 0$), el conjunto $\mathscr{A}$ puede ser caracterizado por: Sean a, b dos números reales fijos tales $0 < a < b < 1$, $\mathscr{A} \subset \{f : [0,a] \cup [b,1] \subset \mathbb{R} \to \mathbb{R}\}$ es el conjunto de aplicaciones de clase C^1, crecientes en $[0,a]$, $f(a) > 1$ y con 0 punto fijo repulsor, decrecientes en $[b,1]$, $f(b) > 1$, $f(1) = 0$ y teniendo al 1 como un único punto crítico, el cual es de orden $\alpha - 1$ para algún $\alpha > 1$. Dos aplicaciones $f, g : [0,a] \cup [b,1] \subset \mathbb{R} \to \mathbb{R}$ son *topológicamente conjugadas* si existe un homeomorfismo $h : I \to I$ tal que $g \circ h = h \circ f$. Decimos que f es *$\mathscr{A}$-structuralmente estable* si existe una C^1-vecindad $\mathscr{U}$ de f en $\mathscr{A}$ tal que cada $g \in \mathscr{U}$ es topológicamente conjugada a f. Es fácil mostrar que si el conjunto maximal invariante de $f \in \mathscr{A}$ y el conjunto maximal invariante de $g \in \mathscr{A}$ no contienen intervalos, entonces f y g son conjugadas.

Sea $f \in \mathscr{A}$ y sea J un intervalo en el maximal invariante contenido en $[0,a] \cup [b,1]$. Dos situaciones se pueden presentar: los intervalos J, $f(J), \cdots$ son disjuntos dos a dos salvo intersección en sus extremos o contrariamente dos de ellos tengan intersección con interior no vació. En el caso que la intersección de dos iterados de J contenga un abierto se muestra que a excepción de un número finito, todo punto de J tiende a un pozo (pues cada $f^n|_J$ es monótona). Decimos que J es un intervalo errante de la aplicación f si los intervalos J, $f(J), \cdots$ son disjuntos dos a dos y el el conjunto $\omega-$límite de J no es igual a una órbita periódica. Nuestro principal objetivo será mostrar la no existencia de pozos ni de intervalos errantes.

El presente trabajo está estructurado en seis capítulos:

En el **Primer Capítulo** damos las definiciones y notaciones básicas. También hacemos una breve discusión bibliográfica referente a la existencia de intervalos errantes y a ciertas caracterizaciones de robustez transitiva como de hiperbolicidad-singular.

En el **Segundo Capítulo** obtenemos que el conjunto de aplicaciones $\mathscr{A}$-estructuralmente estables en $\mathscr{A}$, es no vacío. La motivación es Labarca-Pacifico [34], donde ellos prueban tal propiedad bajo una condición expansiva ($\lambda_s + \lambda_u > 0$). Aquí obtenemos un resultado análogo pero bajo una condición contractiva ($\lambda_s + \lambda_u < 0$).

La prueba en [34] se basa en la inyectividad de las coordenadas de Milnor-Thurston [45], utilizada para construir la conjugación requerida. En nuestro caso, la existencia de un punto crítico hace difícil probar tal inyectividad. Deberemos primero mostrar la no-existencia de pozos ni intervalos errantes. Esto es consecuencia que la imagen del punto crítico 1 es un punto fijo repulsor. Posteriormente, la estabilidad estructural se deriva de los argumentos dados en [34]. Debido a que intervalos errantes complican la comprensión de la dinámica, muchos esfuerzos se han hecho([78], [58], [23], [25], [85], [44], [41], [37], [11], [12], [40], [59], [10], [32], [39], [42] y [79]) para mostrar su no existencia. En todos ellos se requiere de suficiente diferenciabilidad así como otras propiedades de las aplicaciones consideradas. Sólo requiriendo suavidad C^1 o permitiendo puntos críticos llanos, intervalos errantes pueden aparecer (ver, [58], [44], [32], [42]: Teorema 2.3, pág. 43, [15], [25], [26], [73], [10] y [14]). Notar sin embargo que nuestro caso está en el escenario C^1.

En el **Tercer Capítulo** mostramos que si $X \in \mathscr{X}^\infty(M, \partial M)$ exhibe un ciclo singular genérico asociado a una singularidad $\sigma \in \partial M$ con autovalores reales satisfaciendo (2), entonces para k suficientemente grande, X exhibe un conjunto C^k-robusto transitivo conteniendo a σ.

Como consecuencia de este resultado se obtiene que en toda 3-variedad compacta conexa con borde M existe $X \in \mathscr{X}^\infty(M, \partial M)$ exhibiendo un conjunto C^k-robusto transitivo el cual no es hiperbólico-singular para X ni para $-X$. Además este conjunto no es atractor ni repulsor y tiene una singularidad que no es tipo-Lorenz.

En efecto, considere un campo de vectores $X \in \mathscr{X}^\infty(M, \partial M)$ que presenta un ciclo singular asociado a la singularidad disipativa σ cuyos autovalores satisfacen (2). La disipatividad es dada por $\lambda_s + \lambda_u < 0$. Por lo tanto, el teorema nos garantiza la existencia de un conjunto C^k-robusto transitivo que no es hiperbólico-singular, pues el sub-fibrado central no expande volumen debido a la singularidad, y no contrae volumen debido a la órbita periódica O presente en el ciclo. Además por construcción este conjunto no es atractor ni repulsor y tiene una singularidad hiperbólica que no es tipo-Lorenz. Es interesante notar que en nuestra construcción los conjuntos robustos transitivos dejan de ser hiperbólicos-singulares debido a la existencia de una singularidad contractiva, lo que es incompatible con la condición de expansividad del volumen a lo largo del sub-fibrado central.

Estos resultados son conseguidos asumiendo la existencia de coordenadas uniformes C^2-linearizantes en una vecindad de la singularidad σ así como de la aplicación de Poincaré asociada a la órbita periódica O en el ciclo. Con esto obtenemos que la foliación invariante $\mathscr{F}$ y consiguientemente las aplicaciones en $\mathscr{A}$ son diferenciables. Las condiciones que garantizan linearizabilidad son abiertas, pero exigen alta diferenciabilidad del campo, luego esta aproximación no es aplicable al caso C^1, pues como ya se observó la reducción de dimension sólo se puede hacer de manera continua. Es importante resaltar que las C^2-coordenadas linearizantes son requeridas por el hecho que para garantizar la existencia de una foliación de clase C^1 usamos la técnica de transformación de gráfico (ver [6]) en la que se requiere que el campo de vectores sea al menos de clase C^2. Lo mismo sucede si utilizamos el el método de las variedades r-normalmente hiperbólicas (ver [71])

También mostramos que para toda 3-variedad compacta conexa con borde M existe $X \in \mathscr{X}^\infty(M, \partial M)$ tal que para r suficientemente grande, X presenta un conjunto atractor C^r-robusto que no es hiperbólico-singular. Además tal conjunto tiene singularidades que no son tipo-Lorenz. Esto es conseguido con argumentos de Gukenheimer.

En el **Cuarto Capítulo** consideramos el espacio de aplicaciones triangulares $\mathcal{T}$, que corresponde exactamente al conjunto de aplicaciones de Poincaré definidas en una sección transversal Σ asociada al ciclo Γ. Tomando un sistemas de coordenadas $p = (x_p, y_p)$ en Σ, este conjunto de aplicaciones puede ser caracterizado por: Sean $0 < a < b < 1$ dos números reales y U es una vecindad de $[0,1] \times [0,1]$, $\mathcal{T} \subset \{R : [0,1] \times ([0,a] \cup [b,1]) \subset \Sigma \to U\}$, es el conjunto de aplicaciones C^1 satisfaciendo:

a) R es un difeomorfismo en $[0,1] \times ([0,a] \cup [b,1]) \setminus \{y=1\}$). Además, $R(\{y=0\}) \subset [0,1) \times \{0\}$; $R(\{y=a\}) \subset U \backslash ([0,1] \times [0,1])$, $R(\{y=b\}) \subset U \backslash ([0,1] \times [0,1])$; $R(\{y=1\}) = \{(0,c_0)\}$, para algún $c_0 \in (0,1)$.

b) Existe una C^0-foliación (con hojas diferenciables) invariante contractiva $\mathscr{F}$. Además $\{y=0\}$, $\{y=a\}$, $\{y=b\}$ e $\{y=1\}$ son hojas de $\mathscr{F}$.

c) Existen $\alpha = \alpha_R > 1$ y $K > 0$ (independiente de R) tales que

$$y_{R(p)} \leq K \cdot \mid y_p - 1 \mid^{\alpha}, \qquad \forall p \in [0,1] \times [b,1].$$

d) Existen $\tilde{\alpha} = \tilde{\alpha}_R > 1$, $\tilde{K} > 0$ (independiente de R), $0 < \gamma < \frac{1}{2}$, ν, μ (con $1 < \nu \leq \mu$ y $\nu \cdot \mu^{\frac{1-\tilde{\alpha}}{\alpha}} > 1$) y un campo de conos invariante C^{γ} en U transversal a $\mathscr{F}$ tales que

   i) $\| DR(p) \cdot v \| \geq \tilde{K} \cdot \mid y_p - 1 \mid^{\tilde{\alpha}-1} \cdot \| v \|, \quad \forall p \in [0,1] \times [b,1], \forall v \in C^{\gamma}(p).$

   ii) $\nu \cdot \| v \| \leq \| DR(p) \cdot v \| \leq \mu \cdot \| v \|, \quad \forall p \in [0,1] \times [0,a], \forall v \in C^{\gamma}(p).$

Recordemos que la foliación $\mathscr{F}$ es *contractiva* si existen una constante $C > 0$ y $0 < \lambda < 1$ tales que

$$\| DR^n(p) \cdot v \| \leq C \cdot \lambda^n,$$

para todo $n \in \mathbb{N}$, $p \in L$, $L \in \mathscr{F}$ y $v \in T_pL$.

Denotemos por TU el fibrado tangente de U. Dado $p \in U$, $\gamma > 0$, denotamos por $C^{\gamma}(p)$ el γ-cono vertical con inclinación γ, i.e.,

$$C^{\gamma}(p) = \{v \in T_pU : v = (u,w); |u| \leq \gamma \cdot |w|\}.$$

Un campo de conos C^{γ} en U es una aplicación continua $C^{\gamma} : p \in U \mapsto C^{\gamma}(p) \subset T_pU$. C^{γ} es *R-invariante* si para todo $p \in [0,1] \times [0,a] \cup [0,1] \times [b,1]$, $DR(C^{\gamma}(p) \setminus \{0\}) \subset int(C^{\gamma}(R(p)))$. C^{γ} es *transversal* a la foliación $\mathscr{F}$ si para todo $p \in L$ y $L \in \mathscr{F}$, $T_pL \cap C^{\gamma}(p) = \{0\}$. Decimos que una curva ζ es *tangente* al campo de γ-conos C^{γ} si para todo $p \in \zeta$, $T_p\zeta$ está contenido en $C^{\gamma}(p)$.

Nuestro objetivo es mostrar que el conjunto de aplicaciones triangulares que presentan conjuntos C^1-robusto transitivos, es no vacío. En efecto, siguiendo la estrategia utilizada para mostrar el resultado análogo en la clase $\mathscr{A}$(no existencia de intervalos errantes de manera robusta), conseguimos mostrar que el conjunto maximal invariante para un conjunto abierto de aplicaciones $R \in \mathcal{T}$ no admite curvas tangentes al campo de conos C^{γ}. Esto implica que curvas "verticales"que intersectan al maximal invariante tienen iterados con longitudes suficientemente grandes. Los mismos argumentos muestran que conjuntos compactos lejos de $\{y = 1\}$ son hiperbólicos. La transitividad se sigue usando estas dos propiedades argumentando como en la prueba de la robustez del atractor geométrico de Lorenz [24].

En el **Quinto Capítulo** mejoramos lo obtenido en el Tercer Capítulo haciendo uso de las aplicaciones triangulares del cuarto capítulo. Para esto sólo necesitamos diferenciabilidad suficiente para obtener C^1-coordenadas linearizantes. Con esto se obtiene que la foliación invariante $\mathscr{F}$ es sólo continua y consiguientemente la aplicación de hojas es sólo continua. Sin embargo la transformación de Poincaré es una aplicación triangular de acuerdo a lo definido en el cuarto capítulo.

En el **Sexto Capítulo**, mostramos la existencia de un campo de vectores C^{∞} definida sobre una variedad compacta tri-dimensional, el cual exhibe un conjunto atractivo hiperbólico-singular conteniendo una singularidad que no es tipo-Lorenz y un conjunto repulsor hiperbólico no-trivial. La Técnica para la construcción de tal campo de vectores se basa en una combinación con las de Franks-Williams [21] y Morales [49].

Finalmente, es de nuestro interés extender el resultado referido a la existencia de conjuntos robustamente transitivos al caso C^1. En un trabajo aún en desarrollo sólo resta probar que las aplicaciones de primer retorno asociadas a X y Y, para Y C^1 cercano a X, satisfacen las propiedades de las aplicaciones triangulares. La parte dificultosa es encontrar cotas uniformes para las primeras derivadas parciales de la aplicación de primer retorno asociada a todo campo C^1 cercano a X, cuando la dinámica es analizada en una vecindad de la singularidad.

Una pregunta interesante que queda en abierto es saber si la transitividad robusta implica la hiperbolicidad parcial, y si esta permite caracterizar las singularidades en el borde.

Chapter 1

Preliminares

En este capítulo presentamos conceptos y definiciones básicas así como resultados elementales de dinámica discreta como continua en espacios de baja dimensión, que serán utilizados en los capítulos siguientes.

1.1. Sistemas dinámicos sobre una variedad de dimensión uno

1.1.1. Notaciones, definiciones y resultados clásicos

En esta subsección introducimos los conceptos elementales de dinámica unidimensional referente a la existencia de intervalos errantes.

Denotamos por N como el círculo unitario S^1 o el intervalo unitario $I = [0,1]$. Sea $f : N \to N$ una aplicación continua. Para $n \geq 1$, consideramos los iterados f^n, definidos inductivamente por: $f^1 = f$ y $f^{n+1} = f \circ f^n$. Denotamos por f^0 la aplicación identidad. Aquí, $\mid Df(x) \mid$ denota la norma o valor absoluto de la derivada de f en x.

Una aplicación $f : [a,b] \to \mathbb{R}$, con $[a,b] \subset \mathbb{R}$, tiene variación acotada si existe $0 < K < \infty$ tal que para cualquier cubrimiento disjunto I_i de $[a,b]$ la longitud total de $\sum_i |f(I_i)|$ es a lo más K.

Derivada Schwarziana

Definición 1.1.1. *Sea $f : I \to I$ una función de clase C^3 tal que para todo $x \in I$, $Df(x) \neq 0$. Se define la derivada Schwarziana de f, Sf, en $x \in I$ por*

$$Sf(x) = \frac{Df^3(x)}{Df(x)} - \frac{3}{2} \cdot \Big(\frac{Df^2(x)}{Df(x)}\Big)^2.$$

De la definición, se sigue inmediatamente por la regla de la cadena la siguiente fórmula para la derivada Scharziana de la composición de dos funciones f, g de clase C^3:

$$S(g \circ f)(x) = Sg(f(x)) \cdot |Df(x)|^2 + Sf(x).$$

Así, la derivada Schwarziana de los iterados de f es dada por

$$S(f^n)(x) = \sum_{i=1}^{n-1} Sf(f^i(x)) \cdot |Df^i(x)|^2.$$

Por lo tanto, si una aplicación tiene derivada Schwarziana negativa, también lo tienen todos sus iterados.

A continuación se indica una propiedad analítica de aplicaciones con derivada Shwarziana negativa que será utilizada a lo largo del texo.

Lema 1.1.2 (Principio del Mínimo). *Sean T un intervalo cerrado con puntos finales a, b y $f : T \to \mathbb{R}$ una aplicación de clase C^3 con derivada Schwarziana negativa. Si para todo $x \in T$, $Df(x) \neq 0$, entonces*

$$|Df(x)| > \text{mín}\{|Df(a)|, |Df(b)|\}, \forall x \in (a, b).$$

Ver [42], Lema 6.1, pag. 154.

Sea $f : I \to I$ un homeomorfismo. La *órbita* de f de un punto $p \in I$ es el conjunto $O = O_f(p) = \{f^n(p) : n \in \mathbb{Z}\}$.

Sea $O_f(p)$ una órbita de periodo n. Esta órbita es llamada *estable o atractiva* si la *bacía de atracción*,

$$B(p) = \{x : f^{kn}(x) \to p \text{ con} k \to \infty \}$$

de la órbita $O_f(p)$ contiene un intervalo abierto que contiene a p en su clausura. La *bacía inmediata* de $O_f(p)$ es la unión de las componentes conexas de su bacía que contienen un punto de $O_f(p)$.

Sea $f : I \to I$ una aplicación de clase C^1. Un punto periódico p es llamado un *atractor hiperbólico* si $|Df(p)| < 1$, un *repulsor hiperbólico* si $|Df(p)| > 1$ y un *neutral* si $|Df(p)| = 1$. Decimos que c es un *punto crítico* de la aplicación f si $Df(c) = 0$.

Teorema 1.1.3 (Singer [74]). *Si $f : I \to I$ es una aplicación de clase C^3 con derivada Schwarziana negativa entonces*

(i) la bacía inmediata de cualquier órbita periódica atractiva contiene un punto crítico de f o un punto borde del intervalo I;

(ii) cada punto periódico neutral es atractivo;

(iii) no existe intervalo de puntos periódicos.

En particular, el número de órbitas periódicas no-repulsoras es acotado si el número de puntos críticos de f es finito. Más aún, si todos los puntos críticos de f son puntos de retorno entonces f tiene a lo más dos atractores periódicos inescenciales (conteniendo un punto borde de ∂I en su bacía).

Ver, [42], Teorema 6.1, pag 155.

Definición 1.1.4. *Sea $J \subset I$ y sea $f : I \to I$. Decimos que J es un intervalo errante de la aplicación f si:*

1. *los intervalos $J, f(J), ...,$ are disjuntos dos a dos;*

2. *el conjunto $\omega-$límite de J no es igual a una órbita periódica.*

Estabilidad estructural e hiperbolicidad

Denotemos por $C^r(N,N)$, $r \geq 1$, el espacio de los difeomorfismos de N de clase C^r. En $C^r(N,N)$ consideremos la topología C^r definida por la métrica

$$d_r(f,g) = \max_{1\leq i\leq r}\{d(f(x),g(x)), \mid D^i f(x) - D^i g(x) \mid : x \in N\}.$$

Aquí, d es la métrica usual en N. Con la métrica C^r, $C^r(N,N)$ es un espacio métrico completo.

Definición 1.1.5. *Un difeomorfismo $f \in C^r(N,N)$ es estructuralmente estable si existe una vecindad $\mathscr{N}$ de f en $C^r(N,N)$ tal que para cada $g \in \mathscr{N}$, existe un homeomorfismo $h_g : N \to N$ que conjuga f y g, es decir, $g \circ h_g = h_g \circ f$.*

Sea p un punto periódico de periodo n de $f \in C^r(N,N)$. Si $0 \neq |Df^n(p)| \neq 1$ decimos que p es un punto periódico *hiperbólico*. El es un punto periódico *atractivo* si $0 < |Df^n(x)| < 1$ y un punto periódico *repulsivo* si $|Df^n(p)| > 1$. Si $Df^n(p) = 0$ entonces p es llamado un punto periódico *super-atractivo*.

Ahora introduciremos la noción de hiperbolicidad para conjuntos invariantes.

Definición 1.1.6. *Sea $f : N \to N$ una aplicación de clase C^r, $r \geq 1$. Un subconjunto $K \subset N$ es un conjunto hiperbólico repulsivo (o hiperbólico) de f si K es invariante y existen constantes $C > 0$ y $\lambda > 1$ tales que*

$$\mid Df^n(x) \mid > C\lambda^n,$$

para todo $x \in K$ y todo $n \geq 1$.

Lema 1.1.7. *Sea $K \subset N$ un conjunto compacto invariante de una aplicación $f : N \to N$ de clase C^1. Entonces K es un conjunto hiperbólico si y sólo si para cada $x \in K$ existe un entero $n = n(x)$ tal que $\mid Df^n(x) \mid > 1$.*

Ver, [42], Lema 2.1, pag. 220.

Teorema 1.1.8 (Misiurewicz). *Si $f : N \to N$ es una aplicación de clase C^3 con derivada Schwarziana negativa. Un conjunto compacto invariante K es hiperbólico si no contiene puntos criíticos, puntos perioódicos no-hiperbólicos o puntos periódicos atractivos hiperbólicos.*

Ver, [42], Teorema 3.2, pag. 231.

Aplicaciones de Misuirewicz

Sea f una aplicación de clase C^2 y sea $C(f)$ el conjunto de puntos críticos de f, es decir, $C(f) = \{c : Df(c) = 0\}$. Decimos que f satisface la *condición de Misiurewicz* si existe una vecindad W de $C(f)$ tal que

$$\bigcup_{n\geq 1} f^n(C(f)) \cap W = \emptyset.$$

Sea $f : N \to N$ una aplicación de clase C^2. Decimos que un punto crítico c de f es *no-llano* si existe un difeomorfismo local ϕ de clase C^2 con $\phi(c) = 0$ tal que para alguna constante $\alpha \geq 2$, $f(x) = \pm|\phi(x)|^\alpha + f(c)$.

Teorema 1.1.9 (Casi-hiperbolicidad para aplicaciones de Misiurewicz). *Supongamos que $f : N \to N$ es una aplicación de Misiurewicz de clase C^2 con puntos críticos no-llanos y con todos sus puntos periódicos hiperbólicos y repulsores. Entonces para cada vecindad W de $C(f)$ suficientemente pequeña, existen constantes $C > 0$ y $\lambda > 1$ tales que para cada $x \in N$ se tiene lo siguiente:*

(1) si $f^j(x) \notin W$ para $0 \leq j \leq k-1$, entonces

$$|Df^k(x)| \geq C \cdot \lambda^k;$$

(2) Si $f^k(x) \in W$ entonces

$$|Df^k(x)| \geq C \cdot \lambda^k;$$

(3) sin ninguna condición, se tiene

$$|Df^k(x)| \geq C \cdot \lambda^k \cdot \inf_{j=0,\cdots,k-1} |Df(f^j(x))|.$$

Ver, [42], Teorema 6.3, pag. 261.

El próximo teorema muestra que las estimaciones del teorema previo también valen para un conjunto abierto de aplicaciones.

Teorema 1.1.10. *Supongamos que $f : N \to N$ es una aplicación de Misiurewicz de clase C^2 con puntos críticos no-llanos y con todos sus puntos periódicos hiperbólicos y repulsores.*

Entonces existen $C > 0$ y $\lambda > 1$ y una vecindad W de $C(f)$ tal que para cualquier vecindad $U \subset W$ de $C(f)$ existe una vecindad $\mathscr{U}$ de f en la topología C^1 tal que para cada $g \in \mathscr{U}$ se tiene:

(1) si $x, \cdots, g^n \notin W$ para $0 \leq j \leq k-1$, entonces

$$|Dg^n(x)| \geq C \cdot \lambda^n;$$

(2) Si $x, g(x), \cdots, g^{n-1}(x) \notin U$ y $g^n(x) \in W$, entonces

$$|Dg^n(x)| \geq C \cdot \lambda^n;$$

(3) si $g^j(x) \notin U$ para $0 \leq j \leq n-1$ pero no necesariamente $g^n(x) \in W$, entonces

$$|Dg^n(x)| \geq C \cdot \lambda^n \cdot \inf_{j=0,\cdots,n-1} |Dg(g^j(x))|.$$

Ver, [42], Teorema 6.4, pag. 262.

Note que en el Teorema anterior la C^1-vecindad $\mathscr{U}$ depende de la vecindad de los puntos críticos. Así al tomar vecindades arbitrariamente pequeñas de los puntos críticos corremos el riesgo que nuestra vecindad $\mathscr{U}$ se reduzca sólo a la función f. En este trabajo mejoramos el resultado mostrando que la C^1-vecindad no depende de la vecindad del punto crítico (ver Proposición 2.2.9, Capítulo 2).

1.1.2. Discusión bibliográfica sobre existencia de intervalos errantes

En esta subsección vamos a dar una breve exposición referente a la existencia de intervalos errantes para ciertas clases de aplicaciones unidimensionales.

Nuestra exposición relacionada con el problema de la existencia o no-existencia de intervalos errantes es muy cercana a la de Melo, Strien [42] y Lyubich [37].

Sobre no-existencia de Intervalos Errantes:

El Teorema de Poincaré dice que cualquier homeomorfismo del círculo sin puntos periódicos es semi-conjugado a una rotación. Denjoy [16] fortalece este resultado afirmando que está semi-conjungación es de hecho una conjugación siempre que el homeomorfismo y su inverso son suficientemente suaves. La primera prueba de este teorema es debido a Denjoy, quien usa incluso hipótesis más débiles: f es un difeomorfismo de clase C^1 cuya derivada es una función de variación acotada y la segunda prueba es debido a Schwartz [78]. Más precisamente estos resultados son:

Teorema 1.1.11 (Denjoy [16]). *Si $f : S^1 \to S^1$ es un difeomorfismo de clase C^1 y su derivada es una función de variación acotada entonces f no tiene un intervalo errante.*

Ver, [42], Teorema 2.1, pag. 38.

Teorema 1.1.12 (Schwartz [78]). *Sea N un intervalo compacto y $f : N \to N$ una aplicación continua satisfaciendo las siguientes condiciones:*

i) f es monótona por partes de clase C^1 y

ii) la aplicación $x \mapsto log \mid Df(x) \mid$ se extiende a una aplicación Lipschitz sobre N.

Entonces f no tiene intervalos errantes.

Ver, [42], Teorema 3.2, pag. 39.

Como consecuencia se tiene el siguiente corolario.

Corolario 1.1.13. *Sea $f : S^1 \to S^1$ una aplicación de clase C^2 tal que $Df(x) \neq 0$ para todo $x \in S^1$. Entonces f no tiene intervalos errantes.*

Ver, [42], Corolario 1, pag. 41.

Guckenheimer [23] consideró una clase de aplicaciones $\mathscr{C}$ que satisfacen las siguientes propiedades:

1) $f : I \to I$ con $I = [0, 1]$, $f(0) = f(1) = 0$ y $f \in C^3(I)$.

2) f tiene sólo un máximo local $c = c_f$. La función f es estrictamete creciente sobre $[0, c]$ y estrictamente decreciente sobre $[c, 1]$. Además, $D^2 f(c) < 0$.

3) La derivada Scharziana de f es negativa: $Sf(x) < 0$ para todo $x \in I \backslash \{c\}$.

Guckenheimer probó lo siguiente:

Teorema 1.1.14 (Guckenheimer [22]). *Si $f \in \mathscr{C}$ entonces f no tiene intervalos errantes.*

Otros resultados sobre no existencia de intervalos errantes fueron dado por Misiurewicz [47] y [48].

Yoccoz [85] mostró la no existencia de intervalos errantes para homeomorfismos del círculo de clase C^∞ teniendo sólo puntos críticos no-llanos.

de Melo y van Strien [41] y [40] mostraron el mismo resultado para aplicaciones unimodales (no necesariamente teniendo derivada Schwarziana negativa) con puntos críticos no-llanos y también para aplicaciones satisfaciendo la llamada condición de Misiurewicz . Ellos consideraron aplicaciones suaves no-invertibles del intervalo. Más precisamente probaron:

Teorema 1.1.15 (de Melo, van Strien [41], [40]). *Sea $f : [0, 1] \to [0, 1]$ una aplicación de clase C^∞ tal que $f(0) = f(1) = 0$ y f tiene un único punto crítico $c \in (0, 1)$. Si c es no-llano entonces f no tiene intervalos errantes.*

Teorema 1.1.16. (de Melo, van Strien [41], [40]). *Sea $f : N \to N$ un endomorfismo de clase C^3. Supóngase que f satisface las siguientes propiedades: a) todo punto crítico es no-plano; b) la órbita hacia adelante de cada punto crítico no acumula al conjunto de puntos críticos de f. Entonces f no tiene intervalos errantes.*

Lyubich [37] y Blokh, Lyubich [11] mostraron la no-existencia de intervalos errantes para aplicaciones suaves tales que todos sus puntos críticos son puntos de inflexión. Ellos introdujeron nuevas herramientas, las que generalizan a las de Guckenheimer [23] y utilizaron las herramientas analíticas desarrolladas por de Melo y van Strien [41], [40]. En forma más precisa, sea M una variedad compacta unidimensional con borde, es decir, una unión finita disjunta de intervalos y círculos. Consideremos la clase $\mathscr{O}_d$ de aplicaciones $f : M \to M$ de clase C^2, teniendo d, $d \geq 1$, puntos críticos $c_k \in int(M)$ (d-modal"), $1 \leq k \leq d$ y satisfaciendo las siguientes condiciones:

(U1) En una vecindad perforada de los puntos críticos vale la siguiente estimación

$$A_1 \mid x - c_k \mid^{\beta_k} \leq \mid f'(x) \mid \leq A_2 \mid x - c_k \mid^{\beta_k}$$

donde $A_1, A_2, \beta_k > 0$.

(U2) Los puntos críticos c_k son extremos.

(U3) La aplicación f es de clase C^3 y tiene derivada Schwarziana negativa.

Sea $\mathscr{O} = \bigcup_{d=0}^{\infty} \mathscr{O}_d$. Entonces Lyubich [37] obtuvo el siguiente teorema.

Teorema 1.1.17 (Lyubich [37]). *Una apliación $f \in \mathscr{O}$ no tiene intervalos errantes.*

Blokh y Lyubich [11] presentan una continuación de Lyubich [37]. Aquí este resultado se extiende al caso suave. Ellos definen una amplia clase $\mathscr{A}_d$ pero requiriendo las propiedades $(U1)$, $(U2)$ y $(U3)$ pero $(U3)$ sólo es satisfecha localmente en los puntos críticos.

En particular, una aplicación f de clase C^∞ con puntos críticos no-llanos satisface $(U1)$ y $(U3)$ (local).

Sea $\mathscr{A} = \bigcup_{d=0}^{\infty} \mathscr{A}_d$, entonces Blokh y Lyubich [11] probaron el siguiente teorema.

Teorema 1.1.18 (Blokh, Lyubich [11]). *Una apliación $f \in \mathscr{A}$ no tiene intervalos errantes.*

de Melo and van Strien [42] prueban la no existencia de intervalos errantes para aplicaciones no-invertibles de clase C^2 definidas o bien en el intervalo unitario $[0, 1]$ o en círculo unitario S^1 y teniendo un número finito de puntos críticos los cuales son no-llanos (ver [42], Teorema A, pag 267). Ellos en la prueba combinan los instrumentos analíticos desarrollados en de Melo y van Strien [41] con los ingredientes topológicos de Blokh y Lyubich [11]. Puntos de inflexión también se permite en la prueba. Además, la suavidad requerida en la prueba es, precisamente, la misma que necesita en el resultado sobre la no-errante de intervalos errantes, debido a Denjoy [16]. Ellos prueban para hipótesis más generales. Es válida para aplicaciónes en las colecciones $\mathscr{N}^{1+Z}$ y $\mathscr{N}^{1+bv}$, donde, $\mathscr{N}^{1+bv}$ es la clase de aplicaciones absolutamente continuas $f : N \to N$, N tal que la condición $a)$ y $b)$ dadas abajo son satisfechas; y $\mathscr{N}^{1+Z}$ es la clase de aplicaciones absolutamente continuas $f : N \to N$ tal que las condiciones $a)$ and $b')$ dadas abajo son satisfechas.

a) Para cada $x_0 \in N$ existe $\alpha \geq 1$, una vecindad $U(x_0)$ of x_0, y un homeomorfismo $\phi : U(x_0) \to \mathbb{R}$ tal que $\phi(x_0) = 0$ y

$$f(x) = \pm \mid \phi(x) \mid^{\alpha} + f(x_0) \; \forall x \in U(x_0).$$

b) $log(D\phi)$ coincide con la restricción de una función de variación acotada, para cada homeomorfismo ϕ.

b') $\psi = log(D\phi)$ satisface la condición de Zigmund: $\psi(x)+\psi(y)-2\psi(\frac{x+y}{2}) = O(\mid x-y \mid)$, donde $O(t)$ es la aplicación tal que $O(t)/t$ es acotada.

Esta clase incluye, además de aplicaciones de clase C^2 sin puntos críticos llanos, todas las aplicaciones continuas que son C^2 por pedazos (o C^1 y satisfaciendo la condición de Zigmund) y puntos críticos no-llanos. de Melo y van Strien [42] tienen una natural ampliación de las ideas originales de Denjoy para aplicaciones con puntos críticos.

Sobre Existencia de Intervalos Errantes

Denjoy [16], muestra la existencia de aplicaciones C^1 con intervalos errantes. Previamente a enunciar el resultado de Denjoy, vamos a definir un poderoso invariante topológico llamado número de rotación de un homeomorfismo definido sobre S^1, introducido por Poincaré (1881-1886). Sea $f : S^1 \to S^1$ un homeomorfismo del círculo (el cual necesariamente preserva orientación), sin puntos periódicos. Sea $c \in S^1$ y sea $L = [c, f(c))$ un arco positivamente orientado conectando c a $f(c)$. La fracción $\frac{\sharp\{i : f^i(x) \in L, 0 \leq i \leq n-1\}}{n}$ tiene un límite cuando n tiende a infinito y este límite es llamado el *número de rotación* de f.

Teorema 1.1.19 (Denjoy). *Para todo número iracional α, existe un difeomorfismo f de clase C^1 con número de rotación igual a α el cual tiene un intervalo errante.*

Ver, [42], Teorema 2.3, pag. 43.

Permitiendo diferenciabilidad C^1 o existencia de puntos críticos llanos, pueden aparecer intervalos errantes (ver, [58], [15] y [25]). El ejemplo de Ivanov [32], muestra que es esencial asumir que la aplicación tenga alguna suavidad. Más aún Herman [29] muestra ejemplos de $C^{2-\varepsilon}$-difeomorfismos (una aplicación f se dice $C^{2-\varepsilon}$-difeomorfismo si Df satisface la condición de Holder that $\sup_{x \neq y} \frac{|Df(x)-Df(y)|}{|x-y|^{1-\varepsilon}} < \infty$) con número de rotación irracional arbitrario y teniendo un intervalo errante.

1.2. Sistemas dinámicos sobre una variedad de dimensión tres

1.2.1. Definiciones, notaciones y resultados clásicos sobre dinámica discreta

En esta subsección introducimos los conceptos elementales de dinámica tri-dimensional referente a hiperbolicidad, hiperbolicidad-singular, transitividad robusta y atractores robustos.

Conjuntos Hiperbólicos

Sea M una variedad compacta de dimensión n, $n \geq 1$.

Definición 1.2.1. *Sea $f : M \to M$ un difeomorfismo de clase C^r, $r \geq 1$ (o $r = \infty$). Un conjunto compacto $f-$invariante Λ es hiperbólico si existen una descomposición continua del fibrado tangente de M restricto a Λ, $T_\Lambda M$, la cual es Tf invariante:*

$$T_\Lambda M = E^s \oplus E^u; \; Tf(E^s) = E^s; \; Tf(E^u) = E^u;$$

y para la cual existen constantes c positiva y λ con $0 < \lambda < 1$, tales que

$$\begin{aligned} \| Tf^n|_{E^s} \| &< c\lambda^n \; \forall n \geq 0, \\ \| Tf^{-n}|_{E^u} \| &< c\lambda^n \; \forall n \geq 0. \end{aligned}$$

Un punto $x \in M$ es punto α-*límite* de f si existen un punto $y \in M$ y una sucesión de enteros $o < n_1 < n_2 < ...$ tales que $f^{-n_i}(y) \to x$ cuando $i \to \infty$. Similarmente, $x \in M$ es punto $\omega-$límite de f si existen un punto $y \in M$ y una sucesión de enteros $o < n_1 < n_2 < ...$ tales que $f^{n_i}(y) \to x$ cuando $i \to \infty$.

Denotemos por $Clos(A)$ como la cerradura de un conjunto A.

El conjunto de puntos $\alpha-$límite de y es denotado por $\alpha(y)$ y el conjunto de puntos $\omega-$límite de y es denotado por $\omega(y)$.

Sea $L_\alpha(f)$ el conjunto de puntos $\alpha-$límites de f y $L_\omega(f)$ el conjunto de puntos $\omega-$límites de f. Se define

$$L(f) = Clos(L_\alpha(f) \cup L_\omega(f)).$$

Llamamos a $L(f)$ el conjunto límite de f.

$L(f)$ es cerrado e invariante y toda órbita aproxima $L(f)$ en el futuro y pasado. Estudiaremos la situación cuando $L(f)$ es hiperbólico.

Primero recordemos la noción topológica del índice de una aplicación. Sea Δ la bola unitaria cerrada de $\mathbb{R}^n$ y sea $\partial\Delta = S^{n-1}$ la $(n-1)-$esfera. Si $f : \Delta \to \mathbb{R}^n$ es una aplicación continua sin puntos fijos sobre $\partial\Delta$, uno define el índice de f sobre Δ como el grado de la aplicación $x \mapsto \frac{x-f(x)}{|x-f(x)|}$ para $x \in S^{n-1}$. El índice es denotado por $Ind(f, \Delta)$. Si $\phi : \mathbb{R}^n \to \mathbb{R}^n$ es un homeomorfismo que preserva la orientación, $D = \phi(\Delta)$ y f es una aplicación continua de D en $\mathbb{R}^n$ sin puntos fijos sobre ∂D, denotamos por $Ind(f, D) = Ind(\phi^{-1} f \phi, \Delta)$. Llamamos $Ind(f, D)$ el índice de f en D. El índice no depende del homeomorfismo que preserva orientación ϕ.

Es sabido que si $Ind(f, D) \neq 0$ entonces f tiene un punto fijo en D.

La siguiente proposición es probada utilizando el *argumento clásico del índice* (ver Newhouse [56]). Nosotros seguiremos este argumento para mostrar la existencia de puntos fijos para cierto tipo de difeormorfismos del plano (ver la prueba del Teorema 3.1.1, Capítulo 3 y Teorema 4.1.10, Capítulo 4).

Proposición 1.2.2. *Si $L(f)$ es hiperbólico, entonces las órbitas periódicas son densas en $L(f)$.*

1.2.2. Notaciones, definiciones y resultados clásicos sobre dinámica continua

Sea M una variedad compacta diferenciable de dimensión 3 (con borde posiblemente vacío). Denotemos por TM, el *fibrado tangente* de M, el cual está formado por todos los vectores tangentes a M. Un campo de vectores de clase C^r, $r \geq 1$ (o $r = \infty$) es una aplicación de clase C^r $X : M \to TM$ que, a cada punto $p \in M$, asocia un vector

$X(p) \in TM_p$. Denotemos por $\mathscr{X}^r(M)$, $r \geq 1$, como el espacio de los campo de vectores de clase C^r dotado de topología estándar C^r. También vamos a denotar por X_t, $t \in \mathbb{R}$ el flujo generado por X. Recordemos que este flujo es una accción $X : \mathbb{R} \times M \to M$, es decir, $X_0 = id_M$ y $X_s \circ X_t = X_{s+t}$ para todo $s, t \in \mathbb{R}$.

La *órbita* de X para el punto $p \in M$ es el conjunto $O = O_X(p) = \{X_t(x) : t \in \mathbb{R}\}$.

Una *singularidad* de X es un punto $\sigma \in M$ tal que $X(\sigma) = 0$ (equivalentemente, $O_X(\sigma) = \{\sigma\}$). Una *órbita periódica* de X es una órbita $O = O_X(p)$ tal que $X_T(p) = p$ para algún número $T > 0$ mínimo (equivalentemente, O es compacto y $O \neq \{p\}$). Una *órbita cerrada* de X es o bien una singularidad o bien una órbita periódica de X.

El conjunto *omega límite* del punto $p \in M$ es le conjunto $\omega_X(p) = \{x \in M : x = \lim_{n\to\infty} X_{t_n}(p)$, para alguna sucesión $t_n \to \infty\}$. El conjunto *alpha límite* de un punto $p \in M$ es el conjunto $\alpha_X(p) = \{x \in M : x = \lim_{n\to\infty} X_{-t_n}(p)$, para alguna sucesión $t_n \to \infty\}$. Un punto $p \in M$ es llamado *no -errante* para X si para cada $T > 0$ y cada vecindad U de p en M existe $t > T$ tal que $X_t(U) \cap U \neq \emptyset$. El conjunto de puntos no-errantes de X es denotado por $\Omega(X)$.

Un conjunto compacto $\Lambda \subset M$ es:

- Invaviante si $X_t(\Lambda) = \Lambda$, $\forall t \in \mathbb{R}$.
- Transitivo si para algún $p \in \Lambda$, $\Lambda = \omega_X(p)$.
- No-trivial si Λ no es una órbita cerrada de X.
- Aislado si existe una vecindad compacta U de Λ tal que

  $$\Lambda = \bigcap_{t\in\mathbb{R}} X_t(U)$$

  (U es llamado bloque aislante).
- Atractivo si existe una vecindad compacta positivamente invariante U, es decir, $X_t(U) \subset U$, $\forall t \geq 0$, such that

  $$\Lambda = \bigcap_{t\in\mathbb{R}^+} X_t(U)$$
- Atractor si éste es un atractivo transitivo.

Muchos autores llaman atractores a aquellos conjuntos que nosotros llamados de atractivos, ver Milnor [46].

Definición 1.2.3. *Un conjunto aislado y transitivo Λ de X es C^r-robusto transitivo, $r \geq 1$, si existe un bloque aislante U de Λ y una vecindad $\mathscr{U}$ de X (en el espacio $\mathscr{X}^r(M)$) tal que para todo $Y \in \mathscr{U}$, $\bigcap_{t\in\mathbb{R}} Y_t(U)$ es un conjunto transitivo no-trivial de Y.*

Conjuntos Hiperbólicos

Definición 1.2.4. *Un conjunto compacto invariante $\Lambda \subset M$ de X es hiperbólico si existe una descomposición continua y DX_t-invariante del fibrado tangente de M sobre Λ de la forma*

$$TM = E^s \oplus E^X \oplus E^u,$$

tal que para algunas constantes $\lambda, K > 0$ y una métrica riemmaniana en M se tiene:

1. $\| DX_t|E^s \| \leq Ke^{-\lambda t}$, $\forall t > 0$ *(i.e., E^s es el subfibrado $(K,\lambda)-$contractor);*
2. $\| DX_{-t}|E^u \| \leq Ke^{-\lambda t}$, $\forall t > 0$ *(i.e., E^u es el subfibrado $(K,\lambda)-$expansor);*
3. $E^X = \langle X \rangle$ *(i.e., E^X es la dirección del campo).*

Una órbita cerrada de X es hiperbólica si y sólo si vista como un conjunto compacto invariante de X es hiperbólica.

Recordemos algunas propiedades de conjuntos hiperbólicos [31]. Por la teoría de las variedades invariantes tenemos que si $\Lambda \subset M$ es un conjunto hiperbólico para X, entonces para cada $p \in \Lambda$, los conjuntos topológicos

$$W^{ss}(p,X) = \{q \in M : d(X_t(q), X_t(p)) \to 0, \text{cuando}\, t \to \infty\}$$

y

$$W^{uu}(p,X) = \{q \in M : d(X_t(q), X_t(p)) \to 0, \text{cuando}\, t \to -\infty\}.$$

son variedades inmersas unívocamente en M de clase C^r tangentes en p a E^s_p y E^u_p, respectivamente. Estas variedades invariantes se llaman variedades estable e inestable fuertes. Las variedades estable e inestable fuertes locales de tamaño ε son definidas como

$$W^{ss}_\varepsilon(p,X) = \{q \in M : d(X_t(q), X_t(p)) \leq \varepsilon, \forall t \geq 0\}$$

y

$$W^{ss}_{\varepsilon}(p,X) = \{q \in M : d(X_t(q), X_t(p)) \leq \varepsilon, \forall t \leq 0\}.$$

Por lo tanto, otra forma de obtener las variedades estables e inestables fuertes es la siguiente:

$$W^{ss}(p,X) = \bigcup_{t\geq 0} X_{-t}(W^{ss}_{\varepsilon}(X_t(p),X))$$

y

$$W^{uu}(p,X) = \bigcup_{t\geq 0} X_{t}(W^{uu}_{\varepsilon}(X_{-t}(p),X)).$$

Si $p, p' \in \Lambda$, tenemos que $W^{ss}(p,X)$ y $W^{ss}(p',X)$ o son iguales o son disjuntas (similarmente vale para W^{uu}). Las transformaciones $p \in \Lambda \to W^{ss}(p,X)$ y $p \in \Lambda \to W^{uu}(p,X)$ son continuas sobre subconjuntos compactos. Para todo $p \in \Lambda$ definimos

$$W^{s}(p,X) = \bigcup_{t\in\mathbb{R}} (W^{ss}(X_t(p),X)) \text{ } y \text{ } W^{u}(p,X) = \bigcup_{t\in\mathbb{R}} (W^{uu}(X_t(p),X))$$

los conjuntos estables e inestables de la órbita del punto p. En general $W^s(p,X)$ y $W^u(p,X)$ son tangentes en cada $p \in \Lambda$ a $E^s_p \oplus E^X_p$ y $E^X_p \oplus E^u_p$, respectivamente, y dependen continuamente de p.

Conjuntos Hiperbólicos-Singulares

Dado un operador lineal A, definimos $m(A)$ por

$$m(A) = \inf_{v\neq 0} \frac{\| Av \|}{\| v \|}.$$

Definición 1.2.5. *Sea $\Lambda \subset M$ un conjunto compacto invariante de X. Una descomposición continua y DX_t-invariante del fibrado tangente de M sobre Λ, de la forma*

$$T_\Lambda M = E_\Lambda \oplus F_\Lambda,$$

es una descomposición dominada, si existen una métrica Riemanniana en M y constantes positivas K, λ tales que

$$\frac{\| DX_t(x)|E_x \|}{m(DX_t(x)|F_x)} \leq Ke^{-\lambda t}, \ \forall x \in \Lambda, \ \forall t > 0.$$

En este caso se dice que el sub-fibrado F es $(K,\lambda)-$dominado por el subfibrado E.

Definición 1.2.6. *Un conjunto compacto invariante Λ de X es parcialmente hiperbólico, si éste exhibe una descomposición dominada*

$$T_\Lambda M = E^s_\Lambda \oplus E^c_\Lambda$$

en que E^c es dominado por E^s y tal que E^s_Λ es un sub-fibrado contractor, i.e.,

$$\| DX_t(x)|E^s_x \| \leq Ke^{-\lambda t}, \ \forall x \in \Lambda, \ \forall t > 0.$$

El sub-fibrado E^c_Λ es llamado el sub-fibrado central.

Para $x \in \Lambda$ y $t \in \mathbb{R}$ sea $J^c_t(x)$ el valor absoluto del determinante de la transformación lineal

$$DX_t(x)|E^c_x : E^c_x \to E^c_{X_t(x)}.$$

Diremos que un conjunto parcialmente hiperbólico Λ *expande volumen (o $(K,\lambda)-$ expande volumen)* en el sub-fibrado central, si

$$J^c_t(x) \geq K^{-1}e^{\lambda t}, \ \forall x \in \Lambda, \ \forall t > 0.$$

Definición 1.2.7. *Un conjunto compacto invariante Λ de X_t es hiperbólico-singular para X_t si Λ es parcialmente hiperbólico, el sub-fibrado central expande volumen y cada singularidad en Λ es hiperbólica.*

Ciclos Singulares

Un *ciclo* de un campo de vectores X es un subconjunto compacto invariante $\Gamma = \Gamma_c \cup \Gamma_r$ donde

(1) $\Gamma_c = \{\theta_1, ..., \theta_k\}$ es formado por un número finito de elementos críticos (singularidades u órbitas periódicas);

(2) Γ_r consiste de órbitas regulares no-periódicas cuyos conjunto $\alpha-$límite y $\omega-$límite están contenidos en Γ_r;

(3) Para cada $i \in \{1, ..., k-1\}$ existe una órbita γ de Γ_r tal que $\alpha(\gamma) = \theta_i$ y $\omega(\gamma) = \theta_{i+1}$, y para $i = k$, $\theta_{i+1} = \theta_1$.

El ciclo Γ es llamado *singular* si él contiene al menos una singularidad. Es llamado hiperbólico si todos los elementos críticos involucrados en él son hiperbólicos. Un ejemplo de ciclo singular es la Herradura singular [34] en el cual involucra una singularidad y una órbita periódica regular. La existencia the ciclos singulares es un impedimento para la estabilidad estructural y es propia de los campos de vectores. Labarca, Pacifico [34] introducen la Herradura Singular como un modelo para flujos estables no hiperbólicos en el contexto de variedades con borde. Un estudio sistemático de la dinámica asociada a ciclos singulares conteniendo orbitas periódicas fue iniciado por Bamón, Labarca, Mañé, Pacifico [6].

1.2.3. Discusión bibliográfica sobre transitividad robusta, atractores robustos e hiperbolicidad-singular

Atractor de Lorenz

Existe un importante fenómeno, específico del caso continuo, que impide la hiperbolicidad del sistema (y la no densidad de la hiperbolicidad): *acumulación robusta de singularidades por órbitas regulares recurrentes del flujo*. El primer ejemplo de este fenómeno, originado del famoso sistema de ecuaciones diferenciales en $\mathbb{R}^3$,

$$\begin{array}{rcl} \dot{x} & = & -\sigma x + \sigma y \\ \dot{y} & = & \rho x - y - xz \\ \dot{z} & = & xy - \beta x, \end{array} \tag{1.1}$$

fue propuesto por E. Lorenz [36] como un modelo simple para dependencia sensitiva a las condiciones iniciales, vagamente relacionado a la convección de fluidos y predicción del tiempo. Un análisis detallado de estas ecuaciones pueden ser encontrado en el libro de Sparrow [77]. Afraimovich, Bykov, Shil'nikov [2] y Guckenheimer, Williams [GW 79], [W79] proponen modelos geométricos exhibiendo este fenómeno. Un atractor robusto con una singularidad acumulada por órbitas regulares recurrentes del flujo, en el cual las afirmaciones hechas por Lorenz pueden ser verificadas rigurosamente. Una clase abstracta de *conjuntos hiperbólicos generalizados* extendiendo estos modelos geométricos, fue propuesta por Pesin [64] y Sataev [72].

Sin embargo, tomó algunas décadas antes de establecer que atractores similares a los modelos geométricos de Lorenz aparecen en el sistema originalmente propuesto (1.1):

Teorema 1.2.8 (Tucker [82], [83]). *Las ecuaciones de Lorenz soportan un atractor robusto para los parámetros clásicos,* $\sigma = 10$*,* $\rho = 28$ *y* $\beta = 8/3$*.*

A partir de mediados de los años noventa, se estaba elaborando una teoría para explicar el fenómeno de la convivencia robusta de órbitas singulares y regulares en un mismo conjunto transitivo, describiendo las propiedades de tales conjuntos. Un gran progreso fue dado por el trabajo de Morales, Pacifico & Pujals [53], [50], mostrando que conjuntos robustos singulares de flujos 3-dimensionales son atractores o repulsores hiperbólicos-singulares y, que de hecho, ellos comparten las principales propiedades de los atractores geométricos de Lorenz. Para una discusión bibliográfica más detallada sobre Hiperbolicidad singular, atractores robustos y transitividad robusta ver [4] y [13].

Flujos Sobre Variedades con borde

Sistemas dinámicos sobre variedades con borde han sido estudiados por muchos autores (ver por ejemplo, [3], [33], [34], [60], [65], [18], [70], [66], [81], [63], [19]). Pacifico [60] y Labarca, Pacifico [33] extendieron al caso con borde la teoría de sistemas Morse-Smale. En efecto, consideraron un subconjunto abierto-denso $\mathscr{X}_*^\infty(M, \partial M)$ del espacio $\mathscr{X}^\infty(M, \partial M)$ de campo de vectores C^∞ tangente a ∂M, donde M es una variedad compacta con borde ∂M. Ellos extienden la definición de campo de vectores de Morse-Smale a fin de demostrar la equivalencia, en elementos de $\mathscr{X}_*^\infty(M, \partial M)$ con conjunto no-errante simple, entre estabilidad estructural y la condición de ser Morse-Smale.

Con el espíritu de seguir analizando fenónemos que podrían presentarse en flujos definidos en variedad con borde, Labarca & Pacifico [34] construyen un campo de vectores sobre la bola unitaria tri-dimensional tangente al borde, C^1-estructuralmente estable pero con conjunto no-errante no hiperbólico. Esto marca la diferencia del caso sin borde, donde estabilidad C^1-estructural implica hiperbolicidad del conjunto no-errante [28].

El ejemplo construído por Labarca & Pacifico [34] da origen a una aplicación definida sobre un rectángulo en si mismo la cual recuerda a la aplicación *herradura de Smale* [76]. Por esta razón este ejemplo es llamado *herradura singular*.

Transitividad Robusta e Hiperbolicidad-Singular

El teorema de la descomposición espectral de Smale [76], proporciona una descripción del conjunto no-errante de un sistema estructuralmente estable como un número finito de *conjuntos compactos maximales invariantes y transitivos*. Cada una de estas piezas se entiende muy bien, tanto desde el punto de vista determinista y como del estadístico. Además tal descomposición persiste bajo pequeñas perturbaciones C^1. Esto, naturalmente, lleva al estudio de conjuntos transitivos aislados que se mantienen transitivos para todos los sistemas cercanos (robustez transitiva).

A continuación enumeramos una serie de resultados relacionados a: transitividad robusta, atractores e hiperbolicidad-singular para flujos definidos en variedades sin borde.

Teorema 1.2.9 (Doering [17]) *Sea $X \in \mathscr{X}(M^3)$. Si $\Lambda = M^3$ es un conjunto C^1-robusto transitivo de X. Entonces el flujo es de Anosov. En particular el flujo no tiene singularidades.*

En el caso en que Λ es un subconjunto propio de M y contiene singularidades, tenemos la siguiente caracterización.

Teorema 1.2.10 (Molares, Pacifico, Pujals [50]). *Un conjunto C^1-robusto transitivo conteniendo singularidades de un flujo sobre una variedad cerrada de dimensión 3 es un atractor propio o bien un repulsor propio.*

Teorema 1.2.11 (Molares, Pacifico, Pujals [53]). *Un conjunto C^1-robusto transitivo conteniendo singularidades de un flujo X sobre una variedad cerrada de dimensión 3 es hiperbólico-singular para X o bien es hiperbólico-singular para $-X$.*

Las Singularidades de los atractores robustos son tipo-Lorenz

Un conjunto aislado $\Lambda \subset M$ es *robusto singular* para $X \in \mathscr{X}^1(M)$ si existe una vecindad U de Λ en M y una C^1-vecindad $\mathscr{U}$ de X en $\mathscr{X}^1(M)$ tal que para todo $Y \in \mathscr{U}$, $\Lambda_Y(U) = \bigcap_{t\in\mathbb{R}} Y_t(U)$ contiene una singularidad. Denotemos por $S(X)$ como el conjunto de las singularidades de campo de vectores X.

Una singularidad σ de X es *tipo-Lorenz* si los autovalores λ_i, $1 \leq i \leq 3$, de $DX(\sigma)$ son reales y satisfacen $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$.

Teorema 1.2.12 (Molares, Pacifico, Pujals [50]). *Sea Λ un conjunto robusto singular transitivo de $X \in \mathscr{X}^1(M)$. Entonces, para $Y = X$ o bien para $Y = -X$, toda $\sigma \in S(Y) \cap \Lambda$ es tipo-Lorenz y satisface $W^{ss}_Y(\sigma) \cap \Lambda = \{\sigma\}$.*

Como una consecuencia, aplicada a *atractores robustos*, es decir, atractores que persisten para campos de vectores C^1-cercanos y permanecen transitivos, tenemos

Teorema 1.2.13(Molares, Pacifico, Pujals [50]). *Toda singularidad de un atractor robusto sobre una variedad de dimensión 3 cerrada es tipo-Lorenz.*

Atractores Robustos son Hiperbólicos-Singulares

Teorema 1.2.14 (Molares, Pacifico, Pujals [50]). *Atractores C^1-robustos de $X \in \mathscr{X}^1(M^3)$ conteniendo singularidades son conjuntos hiperbólicos-singulares para X.*

Procediendo como en Liao [35] (ver Teorema A), se concluye que atractores robustos *sin singularidades* sobre una 3-variedad son hiperbólicos. Por lo tanto se obtiene la siguiente dicotomía.

Teorema 1.2.15. *Sea Λ un atractor robusto de $X \in \mathscr{X}^1(M^3)$. Entonces Λ es hiperbólico o bien es hiperbólico-singular.*

Consecuencias de Hiperbolicidad-Singular

Proposición 1.2.16. *Sea Λ un conjunto compacto hiperbólico-singular de $X \in \mathscr{X}^1(M)$. Entonces cualquier conjunto compacto invariante $\Gamma \subset \Lambda$ sin singularidades es un conjunto uniformemente hiperbólico.*

Sea Λ un conjunto aislado de $X \in \mathscr{X}^r(M)$, $r \geq 1$. Decimos que Λ es *C^r-robusto periódico* si existe un bloque aislante U de Λ y una vecindad $\mathscr{U}$ de X en $X^r(M)$ tal que para todo $Y \in \mathscr{U}$, $\Lambda_Y(U) = \overline{Per(Y) \cap \Lambda_Y(U)}$.

Teorema 1.2.17 (Molares, Pacifico, Pujals [50]). *Un atractor hiperbólico-singular C^r-robusto periódico $r \geq 1$ sobre una 3-variedad con sólo una singularidad, es C^r-robusto.*

Este resultado, primero probado en [51], da condiciones explícitas *en termino de los flujos cercanos*, suficientes para robustés de atractores. Uno debe aspirar a obtener condiciones suficientes *que sólo dependan del flujo*. Esto fue recientemente logrado por Arroyo y Pujals en [5], donde se obtiene un criterio, que depende sólo del conjunto, para que un conjunto atractor sea C^1-robusto transitivo. Este criterio no tiene restricción sobre el número de singularidades.

Teorema 1.2.18. *Sea $\Lambda = \Lambda_X(U)$ un atractor hiperbólico-singular de $X \in \mathscr{X}^1(M)$ con vecindad aislante U. Entonces el conjunto de puntos periódicos es denso en Λ y Λ es la clase homoclínica de al menos una de estás órbitas. Además, asumiendo que existe $\delta_0 > 0$ tal que para todo $0 < \delta < \delta_0$ el subconjunto maximal invariante positivo de $U \backslash B_\delta(S(X))$ es transitivo. Entonces Λ es C^1-robusto transitivo.*

Atractores y Conjuntos Aislados para Flujos C^1

Sea Λ un conjunto aislado de $X \in \mathscr{X}^r(M)$, $r \geq 1$. Decimos que Λ contiene *robustamente la variedad inestable de un elemento crítico* si existen $x_0 \in C(X) \cap \Lambda$, un bloque aislante U de Λ y una vecindad $\mathscr{U}$ de X en $\mathscr{X}^r(M)$ tal que para todo $Y \in \mathscr{U}$, $W^u_Y(x_0(Y)) \subset U$.

Teorema 1.2.19 (Molares, Pacifico, Pujals [50]). *Sea M una n-variedad compacta, $n \geq 3$ y Λ un conjunto transitivo aislado de $X \in \mathscr{X}^1(M)$. Supónga que todo $x \in C(X) \cap \Lambda$ es hiperbólico. Si Λ contiene robustamente la variedad inestable de un elemento crítico, entonces Λ es un atractor.*

Chapter 2

One-dimensional contracting singular horseshoe

In this chapter we prove C^1 structural stability restrict to certain kind of one-dimensional maps. The motivation is [34] where the authors prove such a property under an expanding condition. Here we obtain analogous result but under a contracting condition resembling the so-called Rovella attractor [71].

2.1. Statement of the Main Theorem

This chapter is motivated by [34] where it is proved C^1 structural stability for certain expanding maps in the interval. Here we obtain an analogous result but under the contracting condition introduced in [71]. Let us give the precise statements of our results.

Hereafter we fix two real numbers a, b such that $0 < a < b < 1$.

Definition 2.1.1.. *Define $\mathscr{A}$ as the set of C^1-maps $f : dom(f) \subset \mathbb{R} \to \mathbb{R}$ satisfying the following properties:*

(i) $dom(f) = [0, a] \cup [b, 1]$. Moreover, f is a increasing on $[0, a]$ and decreasing on $[b, 1]$,

$f(0)=0$ and $|Df(0)|>1$, $f(1)=0$ and $Df(x)=0$ if and only if $x=1$. Additionally $f(a)>1$ and $f(b)>1$.

(ii) There is a constant $\alpha_f>1$ such that

$$f(x) = \mid x-1 \mid^{\alpha_f} H_f(x),\ \forall x \in [b,1]$$

where $H_f:[b,1]\to\mathbb{R}$ is a positive C^1-map with $H_f(1)\neq 0$. Moreover $\lim_{x\to 1}(x-1)\cdot DH_f(x)=0$.

We put the following metric in $\mathscr{A}$:

$$\begin{aligned} d_{C^1}(f,g) &= \max\{\sup_x |f(x)-g(x)|, \sup_x |Df(x)-Dg(x)|,\\ &\quad \sup_x |H_f(x)-H_g(x)|, \sup_x |(x-1)\cdot(DH_f(x)-DH_g(x))|,\\ &\quad |\alpha_f-\alpha_g| : x\in[0,a]\cup[b,1]\}. \end{aligned}$$

Remark 2.1.2.. *For the metric above, a neighborhood $\mathscr{U}$ in $\mathscr{A}$ of a map f is formed by maps belonging to an usual C^1-neighborhood satisfying the additional condition:*

There are constants $K>0$ and $\alpha>1$ such that

$$\frac{1}{K} \mid x-1 \mid^{\alpha-1} \leq \mid Dg(x) \mid \leq K \mid x-1 \mid^{\alpha-1},\ \forall x\in[b,1],\ \forall g\in\mathscr{U}.$$

In relation with this inequality see Lyubich [37].

For all $f\in\mathscr{A}$ there are x_{1f} and x_{2f} with $0<x_{2f}<a<b<x_{2f}<1$ such that $f(x_{1f})=f(x_{2f})=1$. Then, we can define

$$dom_d(f) = [0,x_{1f}]\cup[x_{2f},1].$$

Figure 2.1 displays the essential features of the map $f\in\mathscr{A}$ restricted to $dom_d(f)$.

Definition 2.1.3.. *We say that $f,g\in\mathscr{A}$ are* topologically conjugated *if there is a homeomorphism $h:I\to I$ such that $g\circ h=h\circ f$. We say that f is* structurally stable *if there is a neighborhood $\mathscr{N}$ of f in $(\mathscr{A},d_{C^1})$ such that each $g\in\mathscr{N}$ is topologically conjugate to f.*

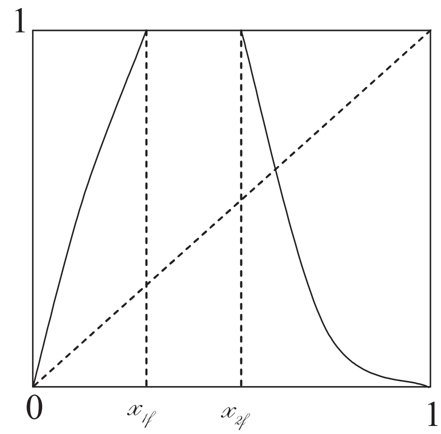

Figura 2.1: f restrict to $dom_d(f)$

Given $f \in \mathscr{A}$ of class C^3 the Schwarzian derivative of f at $x \in dom(f) \setminus \{1\}$ is defined by

$$Sf(x) = \frac{D^3 f(x)}{Df(x)} - \frac{3}{2}\Big(\frac{D^2 f(x)}{Df(x)}\Big)^2.$$

A C^3-map $f \in \mathscr{A}$ has *negative Schwarzian derivative* whenever $Sf(x) < 0$ for all $x \neq 1$. Our main result is the following.

Main Theorem. *Every $f \in \mathscr{A}$ with negative Schwarzian derivative is structurally stable.*

As already mentioned, this result is inspiried by [34] where expanding maps in the interval were considered. The proof in [34] used the injectivity of the Milnor-Thurston coordinates [45] to construct the conjugacy. In our case, the hard part is to prove such an injectivity and to do it we will show the non-existence of wandering intervals first. Afterward the structural stability will follow from the arguments in [34] (see, section 4, pag. 344–345), inspired by Guckenheimer and Williams [24] (see, section 2) and Parry [62] (see, pag 377). Efforts have been made by a number of authors (see, [16], [78], [58], [23], [25], [85], [44], [41], [37], [11], [12], [40], [59], [10], [32], [39], [42] and [79]) towards proving the non-existence of wandering intervals because their appearance complicates the understanding of the dynamics. They involved some smoothness and other ingredients on the considered maps. In the present case we only require C^1-smoothness or allow flat critical point or other ingredients, so wandering intervals may appear (see, [58], [44], [32],

[42]: Theorem 2.3, pag 43, [15], [25], [26], [73], [10] and [14]).

2.2. Proof of the Main Theorem

To achieve our goal we need to recall the Minimum Principle for map with negative Schwarzian derivative ([42], pg. 154).

Lemma 2.2.1. (Minimum Principle). *Let $f \in \mathscr{A}$ be a map with negative Schwarzian derivative and $T = [a_0, b_0]$ be a closed interval contained in $dom(f)$. If $Df(x) \neq 0$ for all $x \in T$, then*

$$| Df(x) | > \text{mín}\{| Df(a_0) |, | Df(b_0) |\}, \qquad \forall x \in (a_0, b_0).$$

The Minimum Principle will be used to find an lower bound, non depending on i, for the derivative $Df^i(x)$ for all x such that $f^i(x)$ is far from 1 and 0. A similar constant was obtained in [55] for the considered maps (see Lemma 3, equation (8), pag. 880).

Lemma 2.2.2.. *Let us consider a map $f \in \mathscr{A}$ with negative Schwarzian derivative and two real numbers c and d, $0 < c < d < 1$. Then there exists a constant $C_0 = C_0(f, c, d) > 0$ such that $\forall i \in \mathbb{N}$ and $x \in dom(f^i)$ with $f^i(x) \in [c, d]$ we have that*

$$|Df^i(x)| \geq C_0. \tag{2.1}$$

Proof. Fix $f \in \mathscr{A}$ with negative Schwarzian derivative and two numbers c and d such that $0 < c < d < 1$.

Define

$$C_0 \;=\; \text{mín}\{c, 1 - d\}.$$

Now, given $i \in \mathbb{N}$ and x such that $f^i(x) \in [c, d]$.

Let us consider $I_x = [\xi_0, \xi_1]$ the maximal interval containing x where f^i is defined. For maximality of I_x we have that either $[0, c] \subset f^i([\xi_0, x])$ and $[d, 1] \subset f^i([x, \xi_1])$, or $[d, 1] \subset f^i([\xi_0, x])$ and $[0, c] \subset f^i([x, \xi_1])$.

In both cases, by Mean Valued Theorem there are $\tilde{\xi_0}, \tilde{\xi_1}$ with $\xi_0 < \tilde{\xi_0} < x < \tilde{\xi_1} < \xi_1$ such that

$$| Df^i(\tilde{\xi_0}) | = \frac{l(f^i([\xi_0, x]))}{l([\xi_0, x])} \geq l(f^i([\xi_0, x])) \geq C_0 \tag{2.2}$$

and

$$| Df^i(\tilde{\xi_1}) | = \frac{l(f^i([x, \xi_1]))}{l([x, \xi_1])} \geq l(f^i([x, \xi_1])) \geq C_0, \tag{2.3}$$

where $l(J)$ denotes the length of the interval J.

Using the Minimum Principle (Lemma 2.2.1), (2.2) and (2.3) we obtain that

$$|Df^i(x)| > \text{mín}\left\{|Df^i(\tilde{\xi_0})|, |Df^i(\tilde{\xi_1})|\right\} \geq C_0,$$

therefore the lemma follows.

Figure 2.2 it illustrates the situation for f^i for $i = 2$. ■

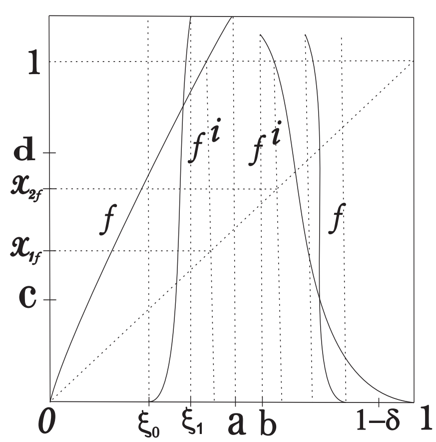

Figura 2.2: Case f^i for $i = 2$.

Now we will use Lemma 2.2.2 to obtain a similar conclusion for maps in a neighborhood of f in $\mathscr{A}$. Indeed, we will prove the following lemma.

Lemma 2.2.3.. *Let us consider $f \in \mathscr{A}$ with negative Schwarzian derivative and two real numbers c_1 and d_1, $0 < c_1 < d_1 < 1$. Then there is $C_1 = C_1(f, c_1, d_1) > 0$ such that for all*

$N \in \mathbb{N}$ there is a C^1-neighborhood $\mathscr{V}_1 = \mathscr{V}_1(f, N, c_1, d_1)$ of f in $\mathscr{A}$ such that for all integers $l \leq N$, $g \in \mathscr{V}_1$ and $x \in dom(g^l)$ such that $g^l(x) \in [c_1, d_1]$ we have that

$$|Dg^l(x)| \geq C_1. \tag{2.4}$$

Proof. Fix $f \in \mathscr{A}$ with negative Schwarzian derivative and c_1 and d_1 as in the statement of Lemma.

Take the real numbers c and d with $0 < c < c_1$ and $d_1 < d < 1$. It follows from the definition of the C^1-topology of $\mathscr{A}$ that for all $j \geq 1$ there is a neighborhood $\bar{\mathscr{V}}(j)$ of f in $\mathscr{A}$ such that if $g \in \bar{\mathscr{V}}(j)$, $w \in dom(g^j)$ and $g^j(w) \in [c_1, d_1]$ then $w \in dom(f^j)$ and

$$f^j(w) \in [c, d]. \tag{2.5}$$

To see this, we extend the functions $g \in \mathscr{A}$ to a endomorphisms at some interval as it is shown in the Figure 2.3.

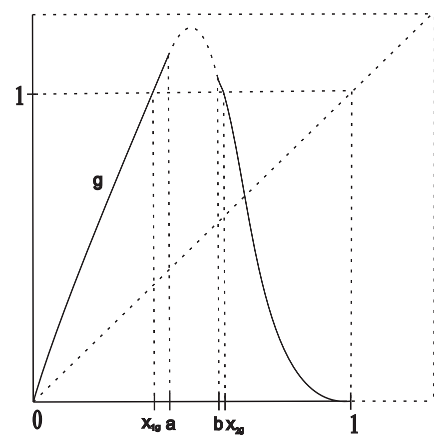

Figura 2.3: Extension of $g \in \mathscr{A}$ to a endomorphisms at the interval.

Take C_0 given by Lemma 2.2.2 applied to f, c and d as above. It follows from the definition of the C^1-topology of $\mathscr{A}$ that for all $i \geq 1$ there is a neighborhood $\tilde{\mathscr{V}}(i)$ of f in $\mathscr{A}$ such that if $g \in \tilde{\mathscr{V}}(i)$, $z \in [0, 1]$ then

$$|Df^i(z) - Dg^i(z)| < \frac{C_0}{2}. \tag{2.6}$$

Now fix an integer $N \geq 1$. Define

$$\bar{\mathscr{U}} = \bar{\mathscr{U}}(f,N) = \bigcap_{1\leq j\leq N} \bar{\mathscr{V}}(j)$$

and

$$\tilde{\mathscr{U}} = \tilde{\mathscr{U}}(f,N) = \bigcap_{1\leq i\leq N} \tilde{\mathscr{V}}(i).$$

Define

$$\mathscr{V}_1 = \bar{\mathscr{U}} \cap \tilde{\mathscr{U}}.$$

Taking $C_1 = \frac{C_0}{2}$, let us prove that the neighborhood $\mathscr{V}_1$ works. For this we fix an integer $1 \leq l \leq N$, $g \in \mathscr{V}_1$ and $x \in dom(g^l)$ such that $g^l(x) \in [c_1, d_1]$. In particular $g \in \bar{\mathscr{U}}$ then (2.5) implies that $f^l(x) \in [c,d]$ (taking $j = l$). Moreover, as in particular $g \in \tilde{\mathscr{U}}$ then (2.6) implies $|Df^l(x) - Dg^l(x)| < \frac{C_0}{2}$ (taking $i = l$). As $f^l(x) \in [c,d]$, (2.1) in Lemma 2.2.2 (taking $i = l$) implies

$$|Df^l(x)| \;\geq\; C_0. \tag{2.7}$$

Therefore, (2.7) implies

$$\begin{aligned} |Dg^l(x)| &= |Dg^l(x) - Df^l(x) + Df^l(x)| \\ &\geq |Df^l(x)| - |Dg^l(x) - Df^l(x)| \\ &\geq C_0 - \frac{C_0}{2} \\ &= \frac{C_0}{2} = C_1, \end{aligned}$$

this completes the proof. ■

Definition 2.2.4.. *Let us consider $f \in \mathscr{A}$. A f-invariant subset $K \subset dom(f)$ is said to be a hyperbolic set for f if there are constants $C > 0$ and $\lambda > 1$ such that for every $x \in K$ and every $n \in \mathbb{N}$*

$$|Df^n(x)| > C\lambda^n.$$

We say that f is hyperbolic in K if K is a hyperbolic set for f.

In an easy way we obtain a following characterization for a compact invariant hyperbolic set. (see [42], see also Lemma 1.1.7 of the Preliminaries).

Proposition 2.2.5.. *Let us consider $f \in \mathscr{A}$ and K an $f-$invariant compact set. Then, K is hyperbolic for f if only if for each $x \in K$ there exists a integer $n = n(x)$ such that $|Df^n(x)| > 1$.*

We say that c is a critical point of $f \in \mathscr{A}$ if $Df(c) = 0$.

Definition 2.2.6.. *Let us consider $f \in \mathscr{A}$ and c a critical point of f. We said that f is hyperbolic far away from c if for any $\delta > 0$, f is hyperbolic in the maximal f-invariant set contained in the complement of a neighborhood of size δ around c.*

First of all we note that if $f \in \mathscr{A}$ with negative Schwarzian derivative then $f \in \mathscr{A}$ is hyperbolic far away from 1. In fact, fix $f \in \mathscr{A}$ with negative Schwarzian derivative. For any $\delta > 0$ let $(1-\delta, 1]$ be a neighborhood of size δ around 1. We define

$$\begin{aligned} V(\delta) &= [0,a] \cup [b, 1-\delta] \\ W_f^k(\delta) &= \left\{x \in V(\delta) | f^i(x) \in V(\delta), i = 0, ..., k-1\right\}, \ \forall k \geq 1 \\ \Lambda_f(\delta) &= \bigcap_{k \geq 1} W_f^k. \end{aligned}$$

By definition, $\Lambda_f(\delta)$ is an forward-invariant compact set that do not contain the critical point 1. By Singer's Theorem (see [42], pag. 155, see also [74]) and Misiurewicz's Theorem (see [42], pag. 231) we have that $\Lambda_f(\delta)$ is a hyperbolic set.

Lemma 2.2.7.. *For every $f \in \mathscr{A}$ with negative Schwarzian derivative there are $\delta_2 = \delta_2(f) > 0$ and a constant $C_2 = C_2(f) > 0$ satisfying the following property: for all $0 < \delta < \delta_2$ there are $\lambda_2 = \lambda_2(f, \delta) > 1$ and a C^1-neighborhood $\mathscr{V}_2 = \mathscr{V}_2(f, \delta)$ of f in $\mathscr{A}$ such that if $g \in \mathscr{V}_2$ and for $k \in \mathbb{N}$, $x \in dom(g^k)$ satisfying that $x, g(x), ..., g^{k-1}(x) \notin (1-\delta, 1]$ and $g^k(x) \in [1-\delta_2, 1]$ then*

$$|Dg^k(x)| \geq C_2 \lambda_2^k. \tag{2.8}$$

Proof. Let consider $f \in \mathscr{A}$ with negative Schwarzian derivative. Choose $\delta_2 > 0$, c_1, d_1 with $0 < c_1 < d_1 < 1$ and a C^1 neighborhood $\bar{\mathscr{V}_2}$ of f in $\mathscr{A}$ in such way that if $g \in \bar{\mathscr{V}_2}$ and $x \in dom(g)$ satisfy $g(x) \geq 1 - \delta_2$ then $x \in [c_1, d_1]$.

Let C_1 be as in Lemma 2.2.3 applied to f, c_1 and d_1 chosen as above. Let consider $\tilde{C}_1 < \text{mín}\{|Df(x)| : x \in [c_1, d_1]\}$. Shrinking $\bar{\mathscr{V}}_2$, we can suppose that for all $g \in \bar{\mathscr{V}}_2$, for all $x \in [c_1, d_1]$ then $|Dg(x)| > \tilde{C}_1$. Define

$$C_2 = \text{mín}\{1, \frac{C_1 \cdot \tilde{C}_1}{2}\}.$$

Now fix $\delta, 0 < \delta < \delta_2$. For such a δ we shall find $\mathscr{V}_2$ and λ_2 as follows:

For all $h \in \mathscr{A}$, we define the auxiliary sets

$$\begin{aligned} V(\delta) &= [0, a] \cup [b, 1 - \delta] \\ W_h^k(\delta) &= \{x \in V(\delta) | h^i(x) \in V(\delta), i = 0, ..., k - 1\}, \ \forall k \geq 1 \\ \Lambda_h(\delta) &= \bigcap_{k \geq 1} W_h^k(\delta). \end{aligned}$$

As we just observed, by Singer's and Misiurewicz's theorems we have that $\Lambda_f(\delta)$ is a hyperbolic set for $f \in \mathscr{A}$ with negative Schwarzian derivative (i.e., f is hyperbolic far away from 1). From this it follows that there are positive constants $\hat{C} = \hat{C}(f, \delta) > 0$ and $\hat{\lambda} = \hat{\lambda}(f, \delta) > 1$ such that for all $k \in \mathbb{N}$ and $x \in \Lambda_f(\delta)$ one has

$$| Df^k(x) | \geq \hat{C}\hat{\lambda}^k.$$

Then, from the openness of the hyperbolicity we can find a C^1-neighborhood $\tilde{\mathscr{V}} = \tilde{\mathscr{V}}(f, \delta)$ of f in $\mathscr{A}$ and constants $\tilde{C} = \tilde{C}(f, \delta)$, $\tilde{\lambda} = \tilde{\lambda}(f, \delta)$, with $\tilde{C} > 0$ and $\hat{\lambda} > \tilde{\lambda} > 1$ such that if $g \in \tilde{\mathscr{V}}$ and x satisfy $g^i(x) \in V(\delta)$ for all $0 \leq i \leq k - 1$, then

$$|Dg^k(x)| \geq \tilde{C}\tilde{\lambda}^k. \tag{2.9}$$

From (2.9) we can find $K = K(f, \delta) \in \mathbb{N}$ and $\hat{\lambda}_2 = \hat{\lambda}_2(f, \delta)$, $\tilde{\lambda} > \hat{\lambda}_2 > 1$ such that if $k \geq K$, $g \in \tilde{\mathscr{V}}$ and x satisfy $g^i(x) \in V(\delta)$ for all $0 \leq i \leq k - 1$, then

$$|Dg^k(x)| \geq \hat{\lambda}_2^k. \tag{2.10}$$

(Just take $K = \text{mín}\{k : \tilde{C}\tilde{\lambda}^k > 1\}$ and $\hat{\lambda}_2$ such that $1 < \hat{\lambda}_2 < \text{mín}\{\tilde{\lambda}\tilde{C}^{\frac{1}{K}}, \tilde{\lambda}\}$).

Let $\mathscr{V}_1$ be the C^1-neighborhood of f in $\mathscr{A}$ given by Lemma 2.2.3 for this K.

Let us consider $\lambda_2 = \lambda_2(f, \delta)$, $1 < \lambda_2 < \hat{\lambda}_2$ such that

$$\lambda_2^K < 2. \tag{2.11}$$

We show that Lemma work with $\mathscr{V}_2 = \mathscr{V}_2(f, \delta) = \bar{\mathscr{V}_2} \cap \tilde{\mathscr{V}} \cap \mathscr{V}_1$ and λ_2 as was chosen.

Fix $g \in \mathscr{V}_2$, $k \in \mathbb{N}$ and $x \in dom(g^k)$ satisfying $x, g(x), ..., g^{k-1}(x) \notin (1-\delta, 1]$ and $g^k(x) \in [1-\delta_2, 1]$.

If $k \geq K$ then (2.10), the definitions of λ_2 and C_2 imply that

$$|Dg^k(x)| \geq \hat{\lambda}_2^k \geq C_2 \lambda_2^k. \tag{2.12}$$

If $k < K$ then (2.4) of Lemma 2.2.3 because $g^{k-1}(x)$ belong to $[c_1, d_1]$, (2.11) and the definitions of λ_2 and C_2 imply that

$$\begin{aligned} |Dg^k(x)| &= |Dg^{k-1}(x)| \cdot |Dg(g^{k-1}(x))| \\ &\geq C_1 \cdot \tilde{C}_1 \\ &= \frac{C_1 \cdot \tilde{C}_1}{2} 2 \\ &> \frac{C_1 \cdot \tilde{C}_1}{2} \lambda_2^K \\ &\geq C_2 \lambda_2^k. \end{aligned} \tag{2.13}$$

Finally, (2.12) and (2.13) imply (2.8) of Lemma 2.2.7. Therefore, the lemma follows. ■

The following Lemma show that points close to the critical point retrieve derivative in a very fast way. The arguments to prove it resembles the ones used in Lemma 1.1. p. 249 in [71].

Lemma 2.2.8.. *For every $f \in \mathscr{A}$ and $0 < C \leq 1$ there are a C^1-neighborhood $\mathscr{V}_3 = \mathscr{V}_3(f, C)$ of f in $\mathscr{A}$, constants $\delta_3 = \delta_3(f, C) > 0$, $\lambda_3 = \lambda_3(f, C) > 1$ and $L = L(f, C) \in \mathbb{N}$ with $C\lambda_3^L > 1$ such that for all $g \in \mathscr{V}_3$, for all $x \in (1-\delta_3, 1)$ there is an integer $l_g(x) > L$ such that, $g^j(x) \notin (1-\delta_3, 1]$ $j = 1, ..., l_g(x) - 1$ and*

$$|Dg^{l_g(x)}(x)| \geq \lambda_3^{l_g(x)}. \tag{2.14}$$

Proof. Fix $f \in \mathscr{A}$ and $0 < C \leq 1$. For every $\eta > 0$ we consider the C^1-neighborhood for f of size η in $\mathscr{A}$, that is

$$\mathscr{V}_\eta = \{g \in \mathscr{A} : d_{C^1}(f,g) < \eta\}.$$

Then

$$\forall z \in [0,a] \cup [b,1], \ \ \forall g \in \mathscr{V}_\eta, \text{ it have } \mid Df(z) \mid -\eta \leq \mid Dg(z) \mid \leq \mid Df(z) \mid +\eta.$$

Moreover, we take ε small enough such that

$$\mid z \mid \leq \varepsilon \Longrightarrow \mid Df(0) \mid -\eta \leq \mid Df(z) \mid \leq \mid Df(0) \mid +\eta.$$

Then $\forall g \in \mathscr{V}_\eta$ and $\mid z \mid \leq \varepsilon$ we have that

$$\mid Df(0) \mid -2\eta \leq \mid Dg(z) \mid \leq \mid Df(0) \mid +2\eta.$$

Put

$$m_\eta = \mid Df(0) \mid -2\eta$$
$$M_\eta = \mid Df(0) \mid +2\eta.$$

Because $Df(0) > 1$ we can choose m, M with $1 < m < Df(0) < M$ such that

$$mM^{\frac{1-\alpha_f}{\alpha_f}} > 1.$$

From this, shrinking η if necessary, because $|\alpha_g - \alpha_f| < \eta$ there is $\hat{\lambda} = \hat{\lambda}(f) > 1$ depending only on f such that for all $g \in \mathscr{V}_\eta$

$$mM^{\frac{1-\alpha_g}{\alpha_g}} > \hat{\lambda} > 1. \tag{2.15}$$

Moreover if η is small enough we have that

$$m < m_\eta < M_\eta < M. \tag{2.16}$$

As $f(1) = 0$ we can choose $0 < \hat{\delta} = \hat{\delta}(f) < 1$ such that $0 < f(x) < \frac{\varepsilon}{2}$ for all $x \in (1-\hat{\delta}, 1)$. Shrinking η again, we can assume that $0 < g(x) < \varepsilon$ for all $x \in (1-\hat{\delta}, 1)$ and for all $g \in \mathscr{V}_\eta$.

For $g \in \mathscr{V}_\eta$ and $x \in [1-\hat{\delta}, 1)$ we define

$$l = l_g(x) = min\{j \geq 1 : g^j(g(x)) \geq \varepsilon\}. \tag{2.17}$$

Note that for all $x \in [1-\hat{\delta}, 1)$, $l_g(x) \geq l_g\left(1-\hat{\delta}\right)$ because of monotonicity of g in the intervals $[0,a]$ and $[b,1]$.

To choose λ_3 we need to make some estimate. Let consider $g \in \mathscr{V}_\eta$ and $x \in (1-\hat{\delta}, 1)$. Denote $z = g(x)$. By definition of $l = l_g(x)$ in (2.17) we have that $z, g(z), ..., g^{l-1}(z) \in [0, \varepsilon)$ and

$$g^l(z) \geq \varepsilon. \tag{2.18}$$

As $g^l(0) = 0$, then (2.16) and the Mean Value Theorem imply that there is $\xi \in (0, z)$ such that

$$g^l(z) = Dg^l(\xi)z \leq M^l z.$$

Replacing this inequality in (2.18) we get

$$M^l z \geq \varepsilon,$$

that is,

$$z \geq \varepsilon M^{-l}. \tag{2.19}$$

By properties of f we have that there exists a positive constant $\hat{K}_f$ such that $\frac{1}{\hat{K}_f} \leq\mid H_f(x) \mid\leq \hat{K}_f$. Using that $|H_f(x) - H_g(x)| < \eta$ we obtain that $|H_g(x)| < \hat{K}_f + \eta$. Moreover, as $g(x) = H_g(x) \mid x-1 \mid^{\alpha_g} = z$ we have that

$$z \leq (\hat{K}_f + \eta)|x-1|^{\alpha_g}.$$

This implies that

$$\mid x-1 \mid^{\alpha_g - 1} \geq \left(\frac{z}{\hat{K}_f + \eta}\right)^{\frac{\alpha_g - 1}{\alpha_g}}. \tag{2.20}$$

From (2.19) and (2.20) imply

$$\begin{aligned} | x-1 |^{\alpha_g-1} &\geq \left(\frac{z}{\hat{K}_f+\eta}\right)^{\frac{\alpha_g-1}{\alpha_g}} \\ &= z^{\frac{\alpha_g-1}{\alpha_g}} \frac{1}{(\hat{K}_f+\eta)^{\frac{\alpha_g-1}{\alpha_g}}} \\ &\geq \left(\frac{\varepsilon}{\hat{K}_f+\eta}\right)^{\frac{\alpha_g-1}{\alpha_g}} \left(M^{\frac{1-\alpha_g}{\alpha_g}}\right)^l. \end{aligned} \tag{2.21}$$

As $H_g(1) \neq 0$ and

$$Dg(x) = |x-1|^{\alpha_g-1}(-\alpha_g H_g(x) + |x-1| DH_g(x)), \tag{2.22}$$

shrinking $\hat{\delta}$ and η, we can choose a positive $\tilde{K}_f$ such that

$$|Dg(x)| \geq \tilde{K}_f |x-1|^{\alpha_g-1}. \tag{2.23}$$

Indeed, take η such that $(\frac{1}{\hat{K}_f}-\eta)\cdot(\alpha_f-\eta)-2\eta > \frac{\alpha_f}{2\cdot\hat{K}_f}$ because $\text{lím}_{x\to 1}(x-1)\cdot DH_f(x) = 0$ we can chose $\hat{\delta}$ small enough such that $|x-1|\cdot|DH_f(x)| < \eta$ for all x such that $|x-1| < \hat{\delta}$. From this and the fact that $|H_f(x) - H_g(x)| < \eta$, $|(x-1)\cdot DH_f(x) - (x-1)\cdot DH_g(x)| < \eta$ and $|\alpha_f - \alpha_g| < \eta$ we get

$$\alpha_g \cdot |H_g(x)| - |x-1| \cdot |DH_g(x)| > \left(\frac{1}{\hat{K}_f} - \eta\right) \cdot (\alpha_f - \eta) - 2 \cdot \eta.$$

Using this last inequality and (2.22) we get

$$| Dg(x) | \geq |x-1|^{\alpha_g-1} \cdot \left(\left(\frac{1}{\hat{K}_f} - \eta\right) \cdot (\alpha_f - \eta) - 2\eta\right).$$

We can take η small enough so that

$$| Dg(x) | \geq \frac{\alpha_f}{2\cdot\hat{K}_f} \cdot |x-1|^{\alpha_g-1}. \tag{2.24}$$

Define $\tilde{K}_f = \frac{\alpha_f}{2\cdot\hat{K}_f}$. Then from (2.24), the proof of (2.23) follows.

From the Chain Rule, (2.23) and (2.16) we have

$$|Dg^l(x)| =| Dg^{l-1}(g(x)) || Dg(x) |\geq \tilde{K}_f|x-1|^{\alpha_g-1}m^{l-1}. \tag{2.25}$$

Note that $\alpha_f - \eta < \alpha_g < \alpha_f + \eta$ implies

$$\alpha_f - 1 - \eta < \alpha_g - 1 < \alpha_f - 1 + \eta \; and \; \frac{1}{\alpha_f+\eta} < \frac{1}{\alpha_g} < \frac{1}{\alpha_f-\eta}.$$

Therefore

$$\frac{\alpha_f - 1 - \eta}{\alpha_f + \eta} < \frac{\alpha_g - 1}{\alpha_g} < \frac{\alpha_f - 1 + \eta}{\alpha_f - \eta}. \tag{2.26}$$

Moreover, (2.25) and (2.21) yields

$$\begin{aligned} |Dg^l(x)| &\geq \tilde{K}_f|x-1|^{\alpha_g-1}m^{l-1} \\ &\geq \tilde{K}_f\left(\frac{\varepsilon}{\hat{K}_f+\eta}\right)^{\frac{\alpha_g-1}{\alpha_g}}\left(M^{\frac{1-\alpha_g}{\alpha_g}}\right)^l m^{l-1} \\ &= \frac{\tilde{K}_f}{m}\left(\frac{\varepsilon}{\hat{K}_f+\eta}\right)^{\frac{\alpha_g-1}{\alpha_g}}\left(mM^{\frac{1-\alpha_g}{\alpha_g}}\right)^l. \end{aligned} \tag{2.27}$$

Using (2.15), (2.26) and (2.27) we get

$$\begin{aligned} |Dg^l(x)| &\geq \frac{\tilde{K}_f}{m}\left(\frac{\varepsilon}{\hat{K}_f+\eta}\right)^{\frac{\alpha_f-1\mp\eta}{\alpha_f\pm\eta}}\hat{\lambda}^l \\ &= C(f)\cdot\hat{\lambda}^l, \end{aligned} \tag{2.28}$$

where the constant

$$C(f) = \frac{\tilde{K}_f}{m}\Big(\frac{\varepsilon}{\hat{K}_f+\eta}\Big)^{\frac{\alpha_f-1-\eta}{\alpha_f+\eta}}$$

if $\frac{\varepsilon}{\hat{K}_f+\eta} \geq 1$ or

$$C(f) = \frac{\tilde{K}_f}{m}\Big(\frac{\varepsilon}{\hat{K}_f+\eta}\Big)^{\frac{\alpha_f-1+\eta}{\alpha_f-\eta}}$$

if $\frac{\varepsilon}{\hat{K}_f+\eta} < 1$.

We claim that for $L_0 \in \mathbb{N}$ fix, there is $\hat{\delta} = \hat{\delta}(L_0)$ such that for all $p \in [0,1] \times [1-\hat{\delta}, 1)$ and for all $R \in \mathscr{V}_\eta$ we obtain $l = l(R,p) > L_0$. Indeed, using (2.19) we obtain that for all $x \in [1-\hat{\delta}, 1)$ and for all $g \in \mathscr{V}_\eta$,

$$l_g(x) \geq l_g\left(1-\hat{\delta}\right) > \frac{ln(\varepsilon) - ln(g(1-\hat{\delta}))}{ln(M)} = L(\hat{\delta}). \tag{2.29}$$

Note that $L(\hat{\delta}) \to \infty$ as $\hat{\delta} \to 0$. Take $\hat{\delta}$ such that $L(\hat{\delta}) > L_0$. Therefore, from (2.29) we have that $l > L_0$ and the proof of claim follows.

Take $L_0 \in \mathbb{N}$ such that $C(f)\hat{\lambda}^{L_0} > 1$. Take $\lambda_3, 1 < \lambda_3 < min\{(C(f))^{\frac{1}{L_0}}\hat{\lambda}, \hat{\lambda}\}$.

From claim give above, (2.28)and definition of λ_3 we obtain

$$\begin{aligned} \mid Dg^l(x) \mid &\geq C(f)\hat{\lambda}^l \\ &= C(f)\hat{\lambda}^{L_0}\hat{\lambda}^{l-L_0} \\ &\geq \lambda_3^{L_0}\lambda_3^{l-L_0} \\ &= {\lambda_3}^l. \end{aligned}$$

Define $L = L(f,C)$ such that $C\lambda_3^L > 1$. From the claim give above we obtain $l > L$ for appropriate choose of $\hat{\delta}$.

The lemma works with $\mathscr{V}_3 = \mathscr{V}_\eta$ for the chosen η and $\delta_3 = \hat{\delta}$. This ends the proof. ■

The following proposition resemble the quasi-hyperbolicity for Misiurewicz maps and its extension to a C^1-neighborhood (see [42], Theorem 6.3, pag. 261 and Theorem 6.4, pag. 262). We improve this result showing that in our case is possible to choose the C^1-neighborhood non depending on the neighborhood of critical point.

Proposition 2.2.9.. *Let us consider $f \in \mathscr{A}$ with negative Schwarzian derivative. Then there are C^1-neighborhood $\mathscr{V}_4 = \mathscr{V}_4(f)$ of f in $\mathscr{A}$ and constants $C_4 = C_4(f) > 0$, $\delta_4 = \delta_4(f) > 0$, $\lambda_4 = \lambda_4(f) > 1$ satisfying the following properties: If $k \in \mathbb{N}$, $g \in \mathscr{V}_4$ and $x \in dom(g^k)$ are such that $g^k(x) \in (1-\delta_4, 1]$, then*

$$|Dg^k(x)| \geq C_4\lambda_4^k. \tag{2.30}$$

Moreover, if $x \in (1-\delta_4, 1)$ then

$$|Dg^k(x)| \geq \lambda_4^k. \tag{2.31}$$

Proof. Fix $f \in \mathscr{A}$ with negative Schwarzian derivative. Let us consider $C_2 > 0$ and δ_2 given in Lemma 2.2.7 applied for f.

Take $C_4 = C_2$. Applying Lemma 2.2.8 for f and $C = C_4$ we obtain a C^1-neighborhood $\mathscr{V}_3$, the real numbers δ_3 and λ_3 and an integer L. Choose δ_4 such that $0 < \delta_4 < \text{mín}\{\delta_2, \delta_3\}$. By the conclusion of Lemma 2.2.7 applied to $\delta = \delta_4$, there are λ_2 and a C^1-neighborhood $\mathscr{V}_2$. Let us consider $\mathscr{V}_4 = \mathscr{V}_2 \cap \mathscr{V}_3$ and choose λ_4 in a such way that $1 < \lambda_4 < \text{mín}\{C_2^{\frac{1}{L}}\lambda_3, \lambda_2\}$. Note that $C_2^{\frac{1}{L}}\lambda_3 > 1$ because $C_2 > C = C_4$.

Now we prove that the proposition works with $\mathscr{V}_4$, C_4, δ_4 and λ_4 as chosen above.

Fix $g \in \mathscr{V}_4$, $k \in \mathbb{N}$ and $x \in dom(g^k)$ and such that $g^k(x) \in (1 - \delta_4, 1]$.

We decompose the orbit $\{g^i(x)\}_{i=0}^k$ in several blocks as follows:

$$\{x = x_1, g(x_1), ..., g^{k_1-1}(x_1)\}, \{y_1 = g^{k_1}(x_1), g(y_1), ..., g^{l_1-1}(y_1)\},$$

$$\{x_2 = g^{l_1}(y_1), g(x_2), ..., g^{k_2-1}(x_2)\}, \{y_2 = g^{k_2}(x_2), g(y_2), ..., g^{l_2-1}(y_2)\}, ...,$$

$$\{x_m = g^{l_{m-1}}(y_{m-1}), g(x_m), ..., g^{k_m}(x_m) = y_m = g^k(x)\},$$

where k_1 is the first integer such that $g^{k_1}(x_1) \in (1-\delta_4, 1)$, $l_1 \geq L$ is given by the conclusion of Lemma 2.2.8 applied to y_1, k_2 is the first integer that $g^{k_2}(x_2) \in (1 - \delta_4, 1)$ and so on.

Notice that $k_1 + l_1 + \cdots + k_{m-1} + l_{m-1} + k_m = k$.

Using the Chain Rule Theorem, (2.8) of Lemma 2.2.7, (2.14) of Lemma 2.2.8, and the definitions of C_4 and λ_4 we obtain

$$\begin{aligned}
|Dg^k(x)| &= |Dg^{k_m}(x_m)|...|Dg^{l_2}(y_2)||Dg^{k_2}(x_2)||Dg^{l_1}(y_1)||Dg^{k_1}(x_1)| \\
&\geq (C_2\lambda_2^{k_m})...\lambda_3^{l_2}(C_2\lambda_2^{k_2})\lambda_3^{l_1}(C_2\lambda_2^{k_1}) \\
&= C_2\lambda_2^{k_1+\cdots+k_m}(C_2\lambda_3^{l_1})...(C_2\lambda_3^{l_{m-1}}) \\
&\geq C_2\lambda_2^{k_1+\cdots+k_m}...\lambda_4^{l_1}...\lambda_4^{l_{m-1}} \\
&\geq C_2\lambda_4^k \\
&\geq C_4\lambda_4^k,
\end{aligned}$$

this proves (2.30) of Proposition 2.2.9.

For finish the proof note that if $x \in (1-\delta_4, 1)$ then in the decomposition of the orbit $\{g^i(x)\}_{i=0}^k$, k_1 no there exists. Therefore following the proof as in the case above we obtain (2.31) of Proposition 2.2.9. Therefore, the proof follows. ■

Corollary 2.2.10.. *Let us consider $f \in \mathscr{A}$ with negative Schwarzian derivative. Then there exists a C^1-neighborhood $\mathscr{V}_5 = \mathscr{V}_5(f)$ of f in $\mathscr{A}$ such that each $g \in \mathscr{V}_5$ is hyperbolic far away from the critical point* 1*.*

Proof. Let us consider $f \in \mathscr{A}$ with negative Schwarzian derivative. Let δ_4 and C^1-neighborhood $\mathscr{V}_4$ given by Proposition 2.2.9. As we just observed before we have that f is hyperbolic for away 1 because $f \in \mathscr{A}$ with negative Schwarzian derivative implies f is hyperbolic far away from the critical point 1, then f is hyperbolic in the maximal f-invariant set in the complement of $(1-\delta_4, 1]$. From the openness of the hyperbolicity we can find a C^1-neighborhood $\tilde{\mathscr{V}_4} = \tilde{\mathscr{V}_4}(f)$ of f in $\mathscr{A}$ such that all $g \in \tilde{\mathscr{V}_4}$ is hyperbolic in the maximal g-invariant set in the complement of $(1-\delta_4, 1]$. Define $\mathscr{V}_5 = \mathscr{V}_4 \cap \tilde{\mathscr{V}_4}$. Now take $g \in \mathscr{V}_5$ and $\delta < \delta_4$. Then, for all x in the maximal g-invariant contained in $[0, 1-\delta]$ we have that either $\forall k > 1, g^k(x) \notin (1-\delta_4, 1]$ or for some $k_1 > 1, g^{k_1}(x) \in (1-\delta_4, 1]$. In the first case, taking k big enough we have that $|Dg^k(x)| > 1$. In the other case, by (2.30) of Proposition 2.2.9, $\mid Dg^{k_1}(x) \mid \geq C_4\lambda_4^{k_1}$.

Now applying the same argument to $g^{k_1}(x)$, we have two alternatives: there is k big enough such that $\mid Dg^{k_1+k}(x) \mid > 1$ or there is k_2 such that $g^{k_2}(g^{k_1}(x)) \in (1-\delta_4, 1)$ in a such case by (2.31) of Proposition 2.2.9 we have that

$$\mid Dg^{k_2+k_1}(x) \mid \geq C_4\lambda_4^{k_1}\lambda_4^{k_2}.$$

Inductively, we obtain that for some k big enough $\mid Dg^k(x) \mid > 1$.

From this and Proposition 2.2.5 we obtain that g is hyperbolic in the maximal g-invariant set in the complement of $(1-\delta, 1]$. As δ is arbitrary then g is hyperbolic far away from the critical point 1. The proof follows. ■

The following theorem it is related with non-existence of wandering intervals. This old problem goes back to Poincaré's work dealing with homeomorphisms of the circle (see [58]). Since then efforts of a number of authors have been directed towards proving the non-existence of wandering intervals because their appearance complicates our understanding of the dynamics.

Theorem 2.2.11.. *Let $f \in \mathscr{A}$ with negative Schwarzian derivative. Then there exists a C^1-neighborhood $\mathscr{V} = \mathscr{V}(f)$ of f in $\mathscr{A}$ such that for all $g \in \mathscr{V}$, the maximal g-invariant set contained in $I = [0,1]$, $\Lambda_g = \bigcap_{i=0}^{\infty} g^{-i}(I)$, has no interval.*

Proof. Fix $f \in \mathscr{A}$ with negative Schwarzian derivative. Let us consider $\mathscr{V}_4$, δ_4 y λ_4 given by Proposition 2.2.9. Take a C^1-neighborhood $\mathscr{V} = \mathscr{V}_5 \subset \mathscr{V}_4$ of f in $\mathscr{A}$ given by Corollary 2.2.10.

Now fix $g \in \mathscr{V}$.

First we observe that g has no sinks. Indeed, by Corollary 2.2.10, g is hyperbolic far away from critical point 1.

Suppose that $\Lambda_g = \bigcap_{i=0}^{\infty} g^{-i}(I)$ contains an interval J. If there are integers $m \neq n$ such that $g^m(J) \cap g^n(J)$ has non empty interior, then g has sinks (see [G 79], Lemma A, pag. 142) and so we get a contradiction. Therefore, the sequence of intervals $\{g^n(J)\}_{n=0}^{\infty}$ are pairwise disjoint and can not accumulate a sink, i.e. J is a wandering interval. From this it follows that $g^n(J)$ accumulate to 1 and

$$| g^n(J) | \to 0 \text{ as } n \to +\infty. \tag{2.32}$$

Let us consider $0 < \eta < \delta_4$ and an integer n_0 in a such way that $\forall n \geq n_0$ $|g^n(J)| < \delta_4 - \eta$. So, if for $n \geq n_0$ $g^n(J) \cap (1-\eta, 1) \neq \emptyset$ then $g^n(J) \subset (1-\delta_4, 1)$.

As the iterates of J accumulates 1, there is a sequence n_k such that $g^{n_k}(J) \subset (1-\delta_4, 1)$. From Chain Rule Theorem we have

$$\begin{aligned} | g^{n_k}(J) | &= | g^{n_k - n_0}(g^{n_0}(J)) | \\ &= | Dg^{n_k - n_0}(\xi) || g^{n_0}(J) |, \end{aligned}$$

for some $\xi \in g^{n_0}(J) \subset (1-\delta_4, 1)$.

Because $g^{n_k - n_0}(\xi) \in g^{n_k}(J) \subset [1-\delta_4, 1]$ we can apply (2.31) to obtain

$$| g^{n_k}(J) | \ \geq \lambda_2^{n_k - n_0} \ | g^{n_0}(J) | \ .$$

As $n_k \to \infty$ we have that

$$| g^{n_k}(J) | \to \infty$$

and so we get a contradiction with (2.32). The proof follows. ■

Proof of the Main Theorem.

Fix $f \in \mathscr{A}$ with negative Schwarzian derivative. Let us consider $\mathscr{U} = \mathscr{V}$ where $\mathscr{V}$ is the C^1-neighborhood of f in $\mathscr{A}$ given by Theorem 2.2.11.

From now on, we follow the classical arguments to prove that two maps on the interval having Markov partitions are conjugates (see by example [34], Lemma 2, pag. 344).

For all $h \in \mathscr{A}$ we denote $J_h = (x_{1h}, x_{2h})$, where x_{1h} and x_{2h} are given in definition of the dynamic domain. Define the sets

$$\Lambda_h = \bigcap_{i=0}^{\infty} h^{-i}(I),$$

and

$$U_h = \bigcup_{i=0}^{\infty} h^{-i}(J_h).$$

Note that $\Lambda_h = I \setminus U_h$.

We denote $I_0^h = [0, x_{1h}]$ and $I_1^h = [x_{2h}, 1]$. We define also $J_0^h = \{x \in I_0^h : h(x) \in J_h\}$ and $J_1^h = \{x \in I_1^h : h(x) \in J_h\}$. Inductively we can define one of the 2^i-components of $h^{-i}(J_h)$ by

$$J^h_{j_1 j_2 \ldots j_i} = \{x \in I_{j_1}, h(x) \in J_{j_2 \ldots j_i}\}$$

where $j_k \in \{0, 1\}$ for $k = 1, 2, ..., i$.

Note that $h(J^h_{j_1 j_2 \cdots j_i}) = J^h_{j_2 \cdots j_i}$ and $h^i(J^h_{j_1 j_2 \cdots j_i}) = J_h$. Moreover $h^{-i}(J_h) = \bigcup_{j_1, \ldots, j_i} J^h_{j_1 j_2 \cdots j_i}$.

Also note that the map $h^i : J^h_{j_1 j_2 \cdots j_i} \to J_h$ is a homeomorphism that will denote by $h_{j_1 j_2 \cdots j_i}$.

Fix $g \in \mathscr{U}$. It follows from Theorem 2.2.11 that the set Λ_g has no interval, equivalently, the set U_g is dense in I.

Let us denote the set Σ_2 of the sequences of $0'$s and $1'$s. Endow Σ_2 with the topology given by the distance

$$d(x,y) = \sum_{i=0}^{\infty} \frac{\mid x_i - y_i \mid}{2^i},$$

where $x, y \in \Sigma_2$ with $x = (x_0, x_1, x_2, x_3, ...)$ and $y = (y_0, y_1, y_2, y_3, ...)$.

The shift map $\sigma : \Sigma_2 \to \Sigma_2$ is defined by $\sigma(x) = y$ where $y_i = x_{i+1}$.

To each point $x \in \Lambda_g$ we can associate a sequence of $0'$s and $1'$s in Σ_2 in the following way: define

$$I_i^g(x) = \begin{cases} 0 & \text{if } g^i(x) \in I_0^g \\ 1 & \text{if } g^i(x) \in I_1^g \end{cases}$$

and let $H_g : \Lambda_g \to \Sigma_2$ be the map given by $H_g(x) = I^g(x)$ where $(I^g(x))_i = I_i^g(x)$. For this map we have the following lemma.

Lemma 2.2.12.. *The map $H_g : \Lambda_g \to \Sigma_2$ as above is a homeomorphism such that $H_g \circ g|_{\Lambda_g} = \sigma \circ H_g$.*

Applying the Lemma 2.2.12 to f and g we can define the homeomorphism $H_{f,g} : \Lambda_f \to \Lambda_g$ given by $H_{f,g}(x) = H_g^{-1}(H_f(x))$ which conjugates the dynamics of f in Λ_f with the dynamic of g in Λ_g.

On the other hand, let $h_0 : J_f \to J_g$ be any orientation preserving homeomorphism. Take $J^f_{j_1 j_2 \dots j_i}$ one of the 2^i-components of $f^{-i}(J_f)$ then the map $h : J^f_{j_1 j_2 \dots j_i} \to J^g_{j_1 j_2 \dots j_i}$ defined by $h(x) = (g_{j_1 j_2 \dots j_i})^{-1} \circ h_0 \circ f^i(x)$ is a orientation preserving homeomorphism. Because $f^i(\partial J^f_{j_1 j_2 \dots j_i}) = \partial J_f$ and $g^i(\partial J^g_{j_1 j_2 \dots j_i}) = \partial J_g$ we note that for $x \in \partial J^f_{j_1 j_2 \dots j_i}$ we have $h(x) = H_{f,g}(x)$. From this we can define the required homeomorphism $\hat{h} : I \to I$:

$$\hat{h}(x) = \begin{cases} H_{f,g}(x) & \text{if } x \in \Lambda_f \\ h(x) & \text{if } x \in U_f. \end{cases}$$

Therefore, by definition 2.1.3, f is C^1 structurally stable (in $\mathscr{A}$); the Main Theorem is proved. ■

Chapter 3

Singular cycles and C^k-robust transitive sets on manifolds with boundary

Let M be a 3-manifold with boundary ∂M. Let X be a C^∞, vector field on M, tangent to ∂M, exhibiting a singular cycle associated to a hyperbolic equilibria $\sigma \in \partial M$ with real eigenvalues $\lambda_{ss} < \lambda_s < 0 < \lambda_u$ satisfying $\lambda_s - \lambda_{ss} - 2\lambda_u > 0$. We prove under generic conditions and k large enough the existence of a C^k robust transitive set of X, that is, any C^k vector field C^k close to X exhibits a transitive set containing σ. In particular, C^∞ vector fields exhibiting C^k robust transitive sets, for k large enough, which are not singular-hyperbolic do exist on any compact 3-manifold with boundary.

3.1. Statement of the Main Theorem

Let M be a compact connected 3-manifold with a possibly nonempty boundary ∂M. We denote by $\mathscr{X}^k(M, \partial M)$, $k \geq 1$, the space of C^k vector fields in M tangent to ∂M (if nonempty) equipped with the standard C^k topology. We fix $X \in \mathscr{X}^k(M, \partial M)$ and denote by X_t, $t \in \mathbb{R}$, the flow generated by X in M. An invariant set Λ of X is *non-trivial* if it is not a single orbit; *transitive* if it is the omega-limit set of one of its points; and *isolated* if there is a compact neighborhood U of it (often called *isolating block*) such that

$$\Lambda = \bigcap_{t\in\mathbb{R}} X_t(U).$$

An isolated set is C^k*-robust transitive* if it exhibits an isolating block U such that for all $Y \in \mathscr{X}^k(M, \partial M)$ C^k-close to X, the continuation of Λ

$$\Lambda_Y = \bigcap_{t\in\mathbb{R}} Y_t(U)$$

is a non-trivial transitive set of Y.

A compact invariant set H of X is *hyperbolic* if its tangent bundle decomposes into an invariant direct sum $E^s_H \oplus E^X_H \oplus E^u_H$ where E^X_H is the subbundle generated by X on H and E^s_H (resp. E^u_H) is contracted (resp. expanded) by the derivative of X_t as $t \to \infty$. The Invariant Manifold Theory [31] then asserts that through any point $x \in H$ passes a strong stable (resp. unstable) manifold $W^{ss}(x)$ (resp. $W^{uu}(x)$) tangent at x to the subspace E^s_x (resp. E^u_x). By saturating these manifolds we obtain the stable and unstable manifolds $W^s(x)$ and $W^u(x)$ of x respectively. A closed orbit is hyperbolic if only if it does as a compact invariant set.

By a *singular cycle* of X we mean a set Γ consisting of a singularity σ and a periodic orbit O (both hyperbolic) and two regular orbits $\gamma_0 \subset W^u(\sigma)\cap W^s(O)$ and $\gamma_1 \subset W^u(O)\cap W^s(\sigma)$. In such a case we say that the singular cycle Γ is *associated* to σ. We shall be mostly interested in singular cycles associated to singularities σ with real eigenvalues $\lambda_u, \lambda_s, \lambda_{ss}$ satisfying the eigenvalue relation

$$\lambda_{ss} < \lambda_s < 0 < \lambda_u \text{ and } \lambda_s - \lambda_{ss} - 2\lambda_u > 0. \tag{3.1}$$

In that case σ is equipped with several invariant manifolds. A two-dimensional stable manifold $W^s(\sigma)$ and a one-dimensional unstable manifold $W^u(\sigma)$. There is also a centre-unstable manifold $W^{cu}(\sigma)$ tangent at σ to the eigenspace associated to $\{\lambda_s, \lambda_u\}$. Centre-unstable manifold is not unique but all of them contain $W^u(\sigma)$ and the tangent sub-bundle $TW^{cu}(\sigma)$ is well defined along $W^u(\sigma)$. Hence we may consider the tangent space $T_qW^{cu}(\sigma)$

at every point $q \in W^u(\sigma)$. We then say that the singular cycle is *simple* if $W^s(O)$ is transversal to $W^{cu}(\sigma)$ along γ_0 and $W^u(O)$ is transverse to $W^s(\sigma)$ along γ_1. This is a *generic* property. Our main result is the following.

Theorem 3.1.1.. *If $X \in \mathscr{X}^\infty(M, \partial M)$ exhibits a generic singular cycle associated to a singularity $\sigma \in \partial M$ with real eigenvalues satisfying (3.1), then for k large enough, X exhibits a C^k-robust transitive set containing σ.*

An application of this result is the following. There is a partial characterization of robust transitive sets on 3-manifolds based on the following definitions. A compact invariant set Λ of X is *partially hyperbolic* if its tangent bundle decomposes into an invariant direct sum $F^s_\Lambda \oplus F^c_\Lambda$ where F^s_Λ is contracting and dominates F^c_Λ. We say that the central subbundle F^c_Λ is volume expanding whenever the flows derivative expands exponentially the volume along F^c_Λ as $t \to \infty$. A *singular-hyperbolic set* is a partially hyperbolic set with hyperbolic equilibria and volume expanding central subbundle. The most representative example of singular-hyperbolic set is the geometric Lorenz attractor [1], [24]. It was proved in [53], [50] that if $\partial M = \emptyset$ then any C^1 robust transitive set containing singularities of X is a proper attractor up to flow reversing (i.e. either for X or $-X$). It is false in the context of boundary-preserving vector fields on 3-manifold with boundary as shows the Singular Horseshoe [34]. Furthermore, in [53], [51] is proved that C^1-robust transitive sets containing singularities are singular hyperbolic sets. An interesting open question is to know if this hold in manifold with boundary. Indeed, the Singular Horseshoe is a singular hyperbolic set. This is not the case in the C^k context. More precisely we have the following corollary:

Corollary 3.1.2.. *For every compact 3-manifold with boundary M there is $X \in \mathscr{X}^\infty(M, \partial M)$ exhibiting a C^k-robust transitive set (for k large enough) which is not singular-hyperbolic, neither up to flow reversing.*

Proof. Take $X \in \mathscr{X}^\infty(M, \partial M)$ exhibiting a generic singular cycle associated to $\sigma \in \partial M$ so that the eigenvalues $\lambda_{ss}, \lambda_u, \lambda_s$ with $\lambda_{ss} < \lambda_s < 0 < \lambda_u$ of σ and $\lambda_s - \lambda_{ss} - 2\lambda_u > 0$ satisfy the additional contracting condition given by the eigenvalues relation $\lambda_u < -\lambda_s$. It follows from Theorem 3.1.1 that X exhibits, for k large enough, a C^k-robust transitive set containing σ. Such a set is not singular-hyperbolic due to the additional eigenvalue relation, which implies that central sub-bundle is not volume expanding at the singularity. In other way, for the reversing flow, also central sub-bundle is not volume expanding because it contains the contracting direction at the orbit O (for $-X$). The proof follows. ∎

Theorem 3.1.3.. *For every compact connected 3-manifold with boundary M there is a vector field $X \in \mathscr{X}^{\infty}(M, \partial M)$ exhibiting for k large enough an attractor C^k-robust set which is not singular-hyperbolic. Moreover such set has singularities which are not Lorenz-like.*

Proof. Take $X \in \mathscr{X}^{\infty}(M, \partial M)$ with a cycle as in figure 3.1. Let assume uniformly C^2-linearizing coordinates for critical elements in the cycle. Suppose that the eigenvalues $\lambda^i_{ss}, \lambda^i_u, \lambda^i_s$ of $\sigma_i \in \partial M$ for $i = 1, 2, 3$ satisfy $\lambda^i_{ss} < \lambda^i_s < 0 < \lambda^i_u$ and satisfy the additional eigenvalues relations $\lambda^0_u < -\lambda^0_s$ and $\left(-\frac{\lambda^j_s}{\lambda^j_u}\right) \cdot \left(-\frac{\lambda^0_s}{\lambda^0_u}\right) < 1$, for $j = 1, 2$. Using the argument of Guckenheimer[23] and Guckenheimer-Williams [24] we can prove that this attractor is C^k-robust, for a such k that permit uniformly C^2-linearizability. The proof follows. ∎

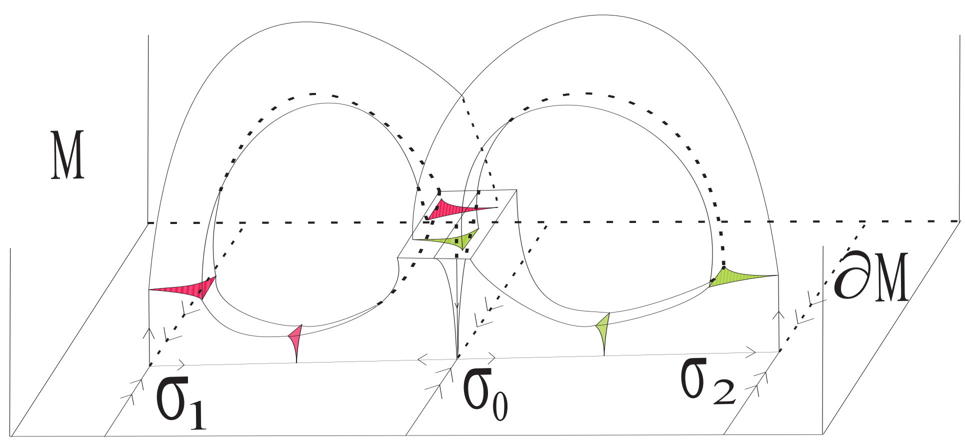

Figura 3.1: Attractor C^k-robust

∎

3.2. One-dimensional analysis

In this section we will consider one dimensional maps that will be associated to the dynamic around the cycle.

3.2.1. Schwarzian derivative

Definition 3.2.1.. *Let $f : Dom(f) \subset \mathbb{R} \to \mathbb{R}$ be a C^3 map. If $Df(x) \neq 0$, the Schwarzian derivative of f at x is defined as*

$$Sf(x) = \frac{D^3 f(x)}{Df(x)} - \frac{3}{2} \cdot \Big(\frac{D^2 f(x)}{Df(x)}\Big)^2.$$

We say that f has negative Schwarzian derivative si $Sf(x) < 0$ for all $x \in Dom(f)$ such that $Df(x) \neq 0$.

From the definition 3.2.1, the following formula for the Schwarzian derivative of the composition of two C^3 maps follows immediately by the chain rule,

$$S(g \circ f)(x) = Sg(f(x)) \cdot \mid Df(x) \mid^2 + Sf(x). \tag{3.2}$$

Hence, the Schwarzian derivative of the iterates of f is given by

$$Sf^n(x) = \sum_{i=0}^{n-1} Sf(f^i(x)) \cdot \mid Df^i(x) \mid^2 .$$

Therefore, if a map has negative Schwarzian derivative, so do all its iterates.

Next we shall state an analytical property of maps with negative Schwarzian derivative that we will be use in this section.

Lemma 3.2.2. (Minimum Principle, see [42], pg. 154). *Let T be a closed interval with end points r and s and $f : T \to \mathbb{R}$ be a map with negative Schwarzian derivative. If $Df(x) \neq 0$ for all $x \in T$, then*

$$\mid Df(x) \mid > \text{mín}\{\mid Df(r) \mid, \mid Df(s) \mid\} \qquad \forall x \in (r, s).$$

Hereafter we fix the unit interval $I = [0, 1]$ and two real numbers a, b such that $0 < a < b < 1$.

Definition 3.2.3. *Define $\mathscr{A}_L$ as the set of C^1 maps $f : [0, a] \cup [b, 1] \subset I \to \mathbb{R}$ satisfying the following properties:*

(i) There exists $\rho_f > 1$ such that $f(x) = \rho_f \cdot x$ for all $x \in [0, a]$. Moreover, f is decreasing on $[b, 1]$, $f(1) = 0$ and $Df(x) = 0$ if and only if $x = 1$. Additionally $f(a) > 1$ and $f(b) > 1$.

(ii) There is a constant $\alpha_f > 1$ such that

$$f(x) = \mid x - 1 \mid^{\alpha_f} \cdot H_f(x), \ \forall x \in [b, 1]$$

where $H_f : [b, 1] \to \mathbb{R}$ is a positive map with $\text{lím}_{x \to 1} H_f(x) \neq 0$ and $\text{lím}_{x\to 1}(x - 1) \cdot DH_f(x) = 0$.

We put the following metric in $\mathscr{A}_L$:

$$\begin{aligned} d_{C^1}(f, g) \quad = \quad & \text{máx}\{\sup_x |f(x) - g(x)|, \sup_x |Df(x) - Dg(x)|, \\ & \sup_x |H_f(x) - H_g(x)|, \sup_x |(x - 1) \cdot (DH_f(x) - DH_g(x))|, \\ & |\alpha_f - \alpha_g| : x \in [0, a] \cup [b, 1]\}. \end{aligned}$$

For all $f \in \mathscr{A}_L$ there are x_{1f} and x_{2f} with $0 < x_{2f} < a < b < x_{2f} < 1$ such that $f(x_{1f}) = f(x_{2f}) = 1$. Then, we can define the dynamic domain

$$dom_d(f) = [0, x_{1f}] \cup [x_{2f}, 1].$$

Figure 3.2 displays the essential features of the map $f \in \mathscr{A}_L$ restricted to $dom_d(f)$.

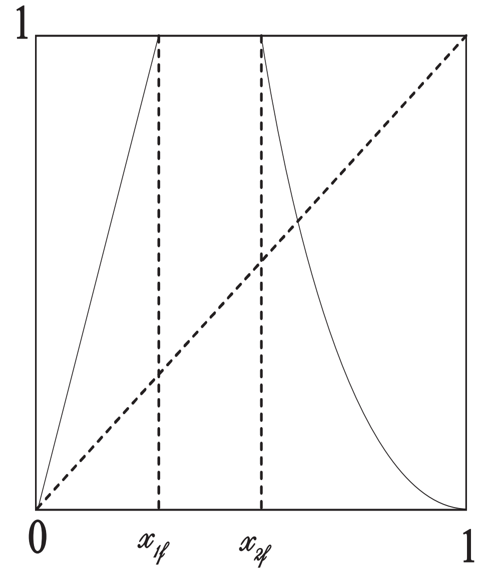

Figura 3.2: $f \in \mathscr{A}_L$ restricted to $dom_d(f)$

Our main result of this section is the following:

Theorem 3.2.4.. *Let consider a C^3 map $f \in \mathscr{A}_L$ such that for all $x \in [b,1), Sf(x) < 0$. Then, there exists a C^1-neighborhood $\mathscr{U} = \mathscr{U}(f)$ of f in $\mathscr{A}_L$ such that for all $g \in \mathscr{U}$, the maximal invariant set*

$$\bigcap_{i=0}^{\infty} g^{-i}(I) \tag{3.3}$$

has no intervals.

We obtain the following result as a direct consequence of Theorem 3.2.4.

Corollary 3.2.5.. *Let us consider $\mathscr{U}$ the C^1-neighborhood given by Theorem 3.2.4. Then for all $g \in \mathscr{V}$ and for all interval J with $J \cap (\bigcap_{i=0}^{\infty} g^{-i}(I)) \neq \emptyset$ there exists $n = n(g,J) \geq 0$ such that $g^n(J) \supset [0,1]$.*

Proof. Fix $g \in \mathscr{V}$ and fix an open interval J such that $J \cap (\bigcap_{i=0}^{\infty} g^{-i}(I)) \neq \emptyset$. For Theorem 3.2.4, the maximal invariant set given by (3.3) don't contain the interval J. Therefore there is a subinterval $\tilde{J} \subseteq J$ with $\tilde{J} = [p,q]$ for some $p,q \in I$ with $p \in J \cap (\bigcap_{i=0}^{\infty} g^{-i}(I)) \neq \emptyset$ and there is an integer $n_0 = n_0(\tilde{J})$ such that $g^{n_0}(q) \in \{x_{1g}, x_{2g}\}$. But $g(\{x_{1g}, x_{2g}\}) = \{1\}$, $g(1) = 0$ and $g(0) = 0$ so there is $n > n_0$ such that $g^n(\tilde{J}) \supseteq [0, 1]$. From this the proof follows. ■

3.2.2. Preliminar lemmas

The Minimum Principle will be used to find a lower bound, non depending on i, for the derivative $|Df^i(x)|$ for all x such that $f^i(x)$ is far from 1. A similar constant was obtained in [55] for the considered maps (see Lemma 3, equation (8), pag. 880).

Lemma 3.2.6..*Let consider a C^3 map $f \in \mathscr{A}_L$ with negative Schwarzian derivative at $[b, 1)$ and two real numbers c and d, $0 < c < d < 1$. Then there exists a constant $C_0 = C_0(f, c, d) > 0$ such that $\forall i \in \mathbb{N}$ and $x \in dom(f^i)$ with $f^i(x) \in [c, d]$ we have that*

$$|Df^i(x)| \geq C_0. \tag{3.4}$$

Proof. Fix a C^3 map $f \in \mathscr{A}_L$ with negative Schwarzian derivative at $[b, 1)$ and two numbers c and d such that $0 < c < d < 1$.

Define

$$C_0 = \text{mín}\{c, 1 - d\}.$$

Let consider a interval $J \subset \mathbb{R}$. We denote by $|J|$ the length of J.

Now, fix $i \in \mathbb{N}$ and x such that $f^i(x) \in [c, d]$.

Let us consider $I_x = [\xi_0, \xi_1]$ the maximal interval containing x where f^i is defined. For maximality of I_x we have that either $[0, c] \subset f^i([\xi_0, x])$ and $[d, 1] \subset f^i([x, \xi_1])$ (if f^i is increasing) or $[d, 1] \subset f^i([\xi_0, x])$ and $[0, c] \subset f^i([x, \xi_1])$ (if f^i is decreasing).

In both cases, by Mean Valued Theorem there are $\tilde{\xi}_0, \tilde{\xi}_1$ with $\xi_0 < \tilde{\xi}_0 < x < \tilde{\xi}_1 < \xi_1$ such that

$$|\ Df^i(\tilde{\xi}_0)\ | = \frac{|f^i([\xi_0, x])|}{|[\xi_0, x]|} \geq |f^i([\xi_0, x])| \geq C_0 \tag{3.5}$$

and

$$| \, Df^i(\tilde{\xi_1}) \, | \quad = \quad \frac{|f^i([x,\xi_1])|}{|[x,\xi_1]|} \geq |f^i([x,\xi_1])| \geq C_0. \tag{3.6}$$

As f restricted to $[0,a]$ is a linear map and $S(f)(x) < 0$ for all $x \in [b,1)$, we can use the Minimum Principle (Lemma 3.2.2), inequalities (3.5) and (3.6) and equality (3.2) to obtain either

$$|Df^i(x)| > \text{mín}\left\{|Df^i(\tilde{\xi_0})|, |Df^i(\tilde{\xi_1})|\right\} \geq C_0,$$

or

$$|Df^i(x)| = \rho_f^i > 1 > C_0.$$

Figure 3.3 it illustrates the situation for f^i with $i = 2$.

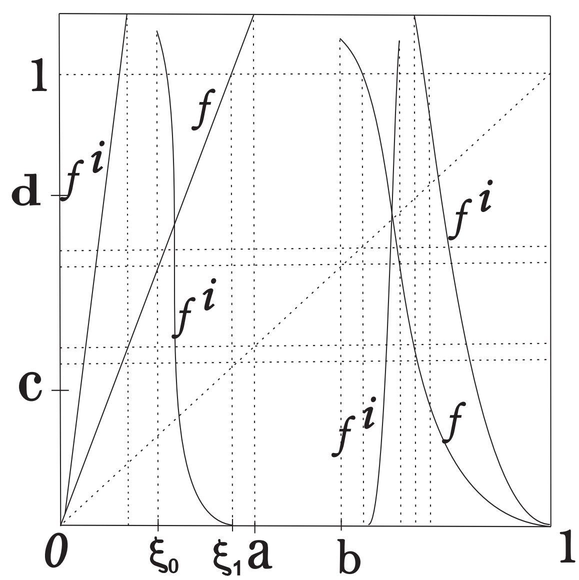

Figura 3.3: Case f^i for $i = 2$.

Therefore the lemma follows. ■

Now we will use Lemma 3.2.6 to obtain a similar conclusion for maps in a neighborhood of f in $\mathscr{A}_L$. Indeed, we will prove the following lemma.

Lemma 3.2.7.. *Let us consider a C^3 map $f \in \mathscr{A}_L$ with negative Schwarzian derivative at $[b,1)$ and two real numbers c_1 and d_1 with $0 < c_1 < d_1 < 1$. Then there is $C_1 = C_1(f, c_1, d_1) > 0$ such that for all $N \in \mathbb{N}$ there is a C^1-neighborhood $\mathscr{V}_1 = \mathscr{V}_1(f, N, c_1, d_1)$ of f in $\mathscr{A}_L$ such that for all integers $l \leq N$, $g \in \mathscr{V}_1$ and $x \in dom(g^l)$ such that $g^l(x) \in [c_1, d_1]$ we have that*

$$|Dg^l(x)| \geq C_1. \tag{3.7}$$

Proof. Fix a C^3 map $f \in \mathscr{A}_L$ with negative Schwarzian derivative at $[b,1)$ and c_1 and d_1 as in the statement of Lemma.

Take the real numbers c and d with $0 < c < c_1$ and $d_1 < d < 1$. It follows from the definition of the C^1-topology in $\mathscr{A}_L$ that for all $j \geq 1$ there is a neighborhood $\bar{\mathscr{V}}(j)$ of f in $\mathscr{A}_L$ such that if $g \in \bar{\mathscr{V}}(j)$, $w \in dom(g^j)$ and $g^j(w) \in [c_1, d_1]$ then $w \in dom(f^j)$ and

$$f^j(w) \in [c, d]. \tag{3.8}$$

To see this, we extend the functions $g \in \mathscr{A}_L$ to a endomorphisms at some interval as it is shown in the Figure 3.4.

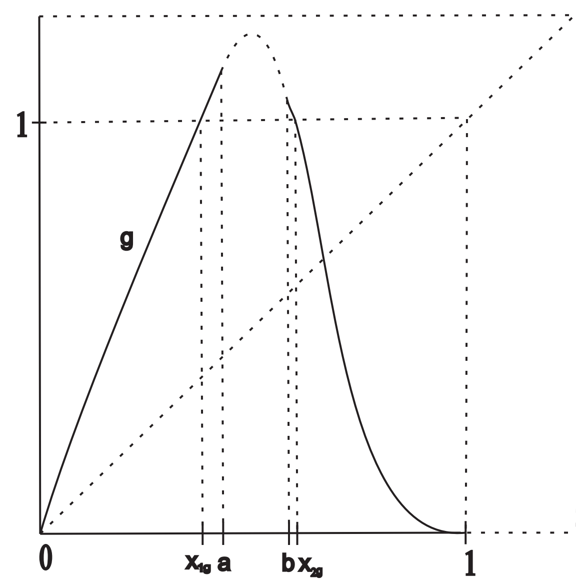

Figura 3.4: Extension of $g \in \mathscr{A}_L$ to a endomorphisms at the interval.

Take C_0 given by Lemma 3.2.6 applied to f, c and d as above. It follows again from the definition of the C^1-topology in $\mathscr{A}_L$ that for all $i \geq 1$ there is a neighborhood $\tilde{\mathscr{V}}(i)$ of f in $\mathscr{A}_L$ such that if $g \in \tilde{\mathscr{V}}(i)$, $z \in [0,1]$ then

$$|Df^i(z) - Dg^i(z)| < \frac{C_0}{2}. \tag{3.9}$$

Now fix an integer $N \geq 1$. Define

$$\bar{\mathscr{U}} = \bar{\mathscr{U}}(f, N) = \bigcap_{1 \leq j \leq N} \bar{\mathscr{V}}(j)$$

and

$$\tilde{\mathscr{U}} = \tilde{\mathscr{U}}(f, N) = \bigcap_{1 \leq i \leq N} \tilde{\mathscr{V}}(i).$$

Define

$$\mathscr{V}_1 = \bar{\mathscr{U}} \cap \tilde{\mathscr{U}}.$$

Taking $C_1 = \frac{C_0}{2}$, let us prove that the neighborhood $\mathscr{V}_1$ works. For this we fix an integer $1 \leq l \leq N$, $g \in \mathscr{V}_1$ and $x \in dom(g^l)$ such that $g^l(x) \in [c_1, d_1]$. In particular $g \in \bar{\mathscr{U}}$ then (3.8) implies that $f^l(x) \in [c, d]$ (taking $j = l$). Moreover, as in particular $g \in \tilde{\mathscr{U}}$ then (3.9) implies $|Df^l(x) - Dg^l(x)| < \frac{C_0}{2}$ (taking $i = l$). As $f^l(x) \in [c, d]$, (3.4) in Lemma 3.2.6 (taking $i = l$) implies

$$|Df^l(x)| \quad \geq \quad C_0. \tag{3.10}$$

Therefore, (3.10) implies

$$\begin{aligned} |Dg^l(x)| &= |Dg^l(x) - Df^l(x) + Df^l(x)| \\ &\geq |Df^l(x)| - |Dg^l(x) - Df^l(x)| \\ &\geq C_0 - \frac{C_0}{2} \\ &= \frac{C_0}{2} = C_1, \end{aligned}$$

this completes the proof. ∎

Definition 3.2.8.. *Let us consider $f \in \mathscr{A}_L$. A f-invariant subset $K \subset dom(f)$ is said to be a hyperbolic set for f if there are constants $C > 0$ and $\lambda > 1$ such that for every $x \in K$ and every $n \in \mathbb{N}$,*

$$|Df^k(x)| > C \cdot \lambda^n.$$

We say that f is hyperbolic in K if K is a hyperbolic set for f.

In an easy way we obtain a following characterization for compact invariant hyperbolic set (see [42]).

Proposition 3.2.9.. *Let us consider $f \in \mathscr{A}_L$ and K an f-invariant compact set. Then, K is hyperbolic for f if only if for each $x \in K$ there exists a integer $n = n(x)$ such that $|Df^n(x)| > 1$.*

We say that c is a critical point of $f \in \mathscr{A}_L$ if $Df(c) = 0$.

Definition 3.2.10.. *Let us consider $f \in \mathscr{A}_L$ and c a critical point of f. We said that f is hyperbolic far away from c if for any $\delta > 0$, f is hyperbolic in the maximal f-invariant set contained in the complement of a neighborhood of size δ around c.*

First of all we note that if a C^3 map $f \in \mathscr{A}_L$ has negative Schwarzian derivative at $[b, 1)$ then $f \in \mathscr{A}_L$ is hyperbolic far away from 1. In fact, fix a C^3 map $f \in \mathscr{A}_L$ with negative Schwarzian derivative at $[b, 1)$. For any $\delta > 0$ let $(1-\delta, 1]$ be a neighborhood of size δ around 1. We define

$$\begin{aligned} V(\delta) &= [0, a] \cup [b, 1-\delta] \\ W_f^k(\delta) &= \left\{x \in V(\delta) | f^i(x) \in V(\delta), i = 0, ..., k-1\right\}, \ \ \forall k \geq 1 \\ \Lambda_f(\delta) &= \bigcap_{k \geq 1} W_f^k. \end{aligned}$$

By definition, $\Lambda_f(\delta)$ is an forward-invariant compact set that do not contain the critical point 1. Then, by Singer's Theorem (see [42], pag. 155, see also [74]) and Misiurewicz's Theorem (see [42], pag. 231) we have that $\Lambda_f(\delta)$ is a hyperbolic set.

Lemma 3.2.11.. *For every C^3 map $f \in \mathscr{A}_L$ with negative Schwarzian derivative at $[b, 1)$ there are $\delta_2 = \delta_2(f) > 0$ and a constant $C_2 = C_2(f) > 0$ satisfying the following property: for all $0 < \delta < \delta_2$ there are $\lambda_2 = \lambda_2(f, \delta) > 1$ and a C^1-neighborhood $\mathscr{V}_2 = \mathscr{V}_2(f, \delta)$ of f in $\mathscr{A}_L$ such that if $g \in \mathscr{V}_2$, $k \in \mathbb{N}$ and $x \in dom(g^k)$ satisfying that $x, g(x), ..., g^{k-1}(x) \notin (1-\delta, 1]$ and $g^k(x) \in [1-\delta_2, 1]$ then*

$$|Dg^k(x)| \geq C_2 \cdot \lambda_2^k. \tag{3.11}$$

Proof. Let consider a C^3 map $f \in \mathscr{A}_L$ with negative Schwarzian derivative at $[b, 1)$. Choose $\delta_2 > 0$ and two real numbers c_1 and d_1 with $0 < c_1 < d_1 < 1$ and a C^1-neighborhood $\bar{\mathscr{V}_2}$ of f in $\mathscr{A}_L$ in such way that if $g \in \bar{\mathscr{V}_2}$ and $x \in dom(g)$ satisfy $g(x) \geq 1 - \delta_2$ then $x \in [c_1, d_1]$.

Let C_1 be as in Lemma 3.2.7 applied to f, c_1 and d_1 chosen as above. Let consider $\tilde{C}_1 < \text{mín}\{|Df(x)| : x \in [c_1, d_1]\}$. Shrinking $\bar{\mathscr{V}_2}$, we can suppose that for all $g \in \bar{\mathscr{V}_2}$ and for all $x \in [c_1, d_1]$ holds $|Dg(x)| > \tilde{C}_1$. Define

$$C_2 = \text{mín}\{1, \frac{C_1 \cdot \tilde{C}_1}{2}\}.$$

Now fix $\delta, 0 < \delta < \delta_2$. For such a δ we shall find $\mathscr{V}_2$ and λ_2 as follows:

For all $h \in \mathscr{A}_L$, we define the auxiliary sets

$$\begin{aligned} V(\delta) &= [0, a] \cup [b, 1 - \delta] \\ W_h^k(\delta) &= \{x \in V(\delta) | h^i(x) \in V(\delta), i = 0, ..., k-1\}, \ \forall k \geq 1 \\ \Lambda_h(\delta) &= \bigcap_{k \geq 1} W_h^k(\delta). \end{aligned}$$

As we just observed (see note below Denifition 3.2.10), for C^3 map $f \in \mathscr{A}_L$ with negative Schwarzian derivative at $[b, 1)$, Singer's and Misiurewicz's theorems give us that $\Lambda_f(\delta)$ is a hyperbolic set (i.e., f is hyperbolic far away from 1). From this it follows that there are positive constants $\hat{C} = \hat{C}(f, \delta) > 0$ and $\hat{\lambda} = \hat{\lambda}(f, \delta) > 1$ such that for all $k \in \mathbb{N}$ and $x \in \Lambda_f(\delta)$ one has

$$| Df^k(x) | \geq \hat{C} \cdot \hat{\lambda}^k.$$

Then, from the openness of the hyperbolicity we can find a C^1-neighborhood $\tilde{\mathscr{V}} = \tilde{\mathscr{V}}(f, \delta)$ of f in $\mathscr{A}_L$ and constants $\tilde{C} = \tilde{C}(f, \delta)$, $\tilde{\lambda} = \tilde{\lambda}(f, \delta)$, with $\tilde{C} > 0$ and $\hat{\lambda} > \tilde{\lambda} > 1$ such that if $g \in \tilde{\mathscr{V}}$ and x satisfy $g^i(x) \in V(\delta)$ for all $0 \leq i \leq k - 1$, then

$$|Dg^k(x)| \geq \tilde{C} \cdot \tilde{\lambda}^k. \tag{3.12}$$

From (3.12) we can find $K = K(f, \delta) \in \mathbb{N}$ and $\hat{\lambda}_2 = \hat{\lambda}_2(f, \delta)$, $\tilde{\lambda} > \hat{\lambda}_2 > 1$ such that if $k \geq K$, $g \in \tilde{\mathscr{V}}$ and x satisfy $g^i(x) \in V(\delta)$ for all $0 \leq i \leq k - 1$, then

$$|Dg^k(x)| \geq \hat{\lambda}_2^k. \tag{3.13}$$

(Just take $K = \text{mín}\{k : \tilde{C} \cdot \tilde{\lambda}^k > 1\}$ and $\hat{\lambda}_2$ such that $1 < \hat{\lambda}_2 < \text{mín}\{\tilde{\lambda} \cdot \tilde{C}^{\frac{1}{K}}, \tilde{\lambda}\}$).

Let $\mathscr{V}_1$ be the C^1-neighborhood of f in $\mathscr{A}_L$ given by Lemma 3.2.7 for this K.

Let us consider $\lambda_2 = \lambda_2(f, \delta)$, $1 < \lambda_2 < \hat{\lambda}_2$ such that

$$\lambda_2^K < 2. \tag{3.14}$$

We show that Lemma works with $\mathscr{V}_2 = \mathscr{V}_2(f, \delta) = \bar{\mathscr{V}_2} \cap \tilde{\mathscr{V}} \cap \mathscr{V}_1$ and this choosing of λ_2.

Fix $g \in \mathscr{V}_2$, $k \in \mathbb{N}$ and $x \in dom(g^k)$ satisfying $x, g(x), ..., g^{k-1}(x) \notin (1 - \delta, 1]$ and $g^k(x) \in [1 - \delta_2, 1]$.

If $k \geq K$ then (3.13), the choosing of λ_2 and C_2 imply that

$$|Dg^k(x)| \geq \hat{\lambda}_2^k \geq C_2 \cdot \lambda_2^k. \tag{3.15}$$

If $k < K$ then (3.7) of Lemma 3.2.7 (because $g^{k-1}(x)$ belong to $[c_1, d_1]$), (3.14) and the choosing of λ_2 and C_2 imply that

$$\begin{aligned} |Dg^k(x)| &= |Dg^{k-1}(x)| \cdot |Dg(g^{k-1}(x))| \\ &\geq C_1 \cdot \tilde{C}_1 \\ &= \frac{C_1 \cdot \tilde{C}_1}{2} \cdot 2 \\ &> \frac{C_1 \cdot \tilde{C}_1}{2} \cdot \lambda_2^K \\ &\geq C_2 \cdot \lambda_2^k. \end{aligned} \tag{3.16}$$

Finally, (3.15) and (3.16) imply (3.11) of Lemma 3.2.11. Therefore, the lemma follows.
■

The following Lemma show that points close to the critical point retrieve derivative in a very fast way. The arguments to prove it resembles the ones used in Lemma 1.1. p. 249 in [71].

Lemma 3.2.12.. *For every $f \in \mathscr{A}_L$ and $0 < C \leq 1$ there are a C^1-neighborhood $\mathscr{V}_3 = \mathscr{V}_3(f, C)$ of f in $\mathscr{A}_L$, constants $\delta_3 = \delta_3(f, C) > 0$, $\lambda_3 = \lambda_3(f, C) > 1$ and $L = L(f, C) \in \mathbb{N}$*

with $C \cdot \lambda_3^L > 1$ such that for all $g \in \mathscr{V}_3$, for all $x \in (1-\delta_3, 1)$ there is an integer $l_g(x) > L$ such that, $g^j(x) \notin (1-\delta_3, 1]$ $j = 1, ..., l_g(x) - 1$ and

$$|Dg^{l_g(x)}(x)| \geq \lambda_3^{l_g(x)}. \tag{3.17}$$

Proof. Fix $f \in \mathscr{A}_L$ and $0 < C \leq 1$. For every $\eta > 0$ we consider the C^1-neighborhood for f of size η in $\mathscr{A}_L$, that is

$$\mathscr{V}_\eta = \{g \in \mathscr{A}_L : d_{C^1}(f, g) < \eta\}.$$

Then $\forall z \in [0, a] \cup [b, 1]$, $\forall g \in \mathscr{V}_\eta$ it have

$$| Df(z) | - \eta \leq | Dg(z) | \leq | Df(z) | + \eta.$$

In particular, $\forall g \in \mathscr{V}_\eta$ and $z \in [0, a]$ we have that

$$\rho_f - \eta \leq \rho_g \leq \rho_f + \eta,$$

where $Df(z) = \rho_f$ and $Dg(z) = \rho_g$.

Because $\rho_f > 1$ we can choose m, M with $1 < m < \rho_f < M$ such that

$$m \cdot M^{\frac{1-\alpha_f}{\alpha_f}} > 1.$$

From this, shrinking η if necessary, because α_g is close to α_f there is $\hat{\lambda} = \hat{\lambda}(f) > 1$, depending only on f, such that for all $g \in \mathscr{V}_\eta$

$$m \cdot M^{\frac{1-\alpha_g}{\alpha_g}} > \hat{\lambda} > 1. \tag{3.18}$$

Moreover η is chosen such that

$$m < \rho_f - \eta < \rho_f + \eta < M. \tag{3.19}$$

As $f(1) = 0$ we can choose $0 < \hat{\delta} = \hat{\delta}(f) < 1$ such that $0 < f(x) < \frac{a}{2}$ for all $x \in [1-\hat{\delta}, 1)$. Shrinking η again, we can assume that $0 < g(x) < a$ for all $g \in \mathscr{V}_\eta$ and $x \in [1-\hat{\delta}, 1)$.

For $g \in \mathscr{V}_\eta$ and $x \in [1-\hat{\delta}, 1)$ we define

$$l = l_g(x) = min\{j \geq 1 : g^j(g(x)) \geq a\}. \tag{3.20}$$

Note that for all $x \in [1-\hat{\delta}, 1)$,

$$l_g(x) \geq l_g\left(1-\hat{\delta}\right) \tag{3.21}$$

because of monotonicity of g in the intervals $[0,a]$ and $[b,1]$.

To choose λ_3 we need to make some estimative. Let consider $g \in \mathscr{V}_\eta$ and $x \in [1-\hat{\delta}, 1)$. Denote $z = g(x)$. By definition of $l = l_g(x)$ in (3.20) we have that $z, g(z), ..., g^{l-1}(z) \in [0,a)$ and

$$M^l \cdot z \geq g^l(z) = (\rho_g)^l \cdot z \geq a. \tag{3.22}$$

so

$$z \geq a \cdot M^{-l}. \tag{3.23}$$

Moreover, by properties of f we have that there exists a positive constant $\hat{K}_f$ such that $\frac{1}{\hat{K}_f} \leq \mid H_f(x) \mid \leq \hat{K}_f$. Using that $|H_f(x) - H_g(x)| < \eta$ we obtain that $\frac{1}{\hat{K}_f} - \eta < |H_g(x)| < \hat{K}_f + \eta$.

Furthermore, as $g(x) = H_g(x) \mid x-1 \mid^{\alpha_g} = z$ we have that

$$z \leq (\hat{K}_f + \eta)|x-1|^{\alpha_g}.$$

This implies that

$$\mid x-1 \mid^{\alpha_g - 1} \geq \left(\frac{z}{\hat{K}_f + \eta}\right)^{\frac{\alpha_g - 1}{\alpha_g}}. \tag{3.24}$$

From (3.23) and (3.24) we obtain

$$\mid x-1 \mid^{\alpha_g - 1} \quad \geq \quad \left(\frac{a}{\hat{K}_f + \eta}\right)^{\frac{\alpha_g - 1}{\alpha_g}} \cdot \left(M^{\frac{1-\alpha_g}{\alpha_g}}\right)^l. \tag{3.25}$$

From Chain Rule Theorem we obtain

$$|Dg(x)| \geq |x-1|^{\alpha_g} \cdot (\alpha_g \cdot |H_g(x)| - |x-1| \cdot |DH_g(x)|) \tag{3.26}$$

Moreover, because $\lim_{x\to 1}(x-1)\cdot DH_f(x) = 0$ we can chose $\hat{\delta}$ small enough such that $|x-1|\cdot|DH_f(x)| < \eta$ for all x such that $|x-1| < \hat{\delta}$. From this and the fact that $|H_f(x) - H_g(x)| < \eta$, $|(x-1)\cdot DH_f(x) - (x-1)\cdot DH_g(x)| < \eta$ and $|\alpha_f - \alpha_g| < \eta$ we get

$$\alpha_g \cdot |H_g(x)| - |x-1|\cdot|DH_g(x)| > \Big(\Big(\frac{1}{\hat{K}_f} - \eta\Big)\cdot(\alpha_f - \eta) - 2\cdot\eta\Big).$$

Using this last inequality and (3.26) we get

$$| \; Dg(x) \; | \geq |x-1|^{\alpha_g - 1}\cdot\Big(\Big(\frac{1}{\hat{K}_f} - \eta\Big)\cdot(\alpha_f - \eta) - 2\eta\Big).$$

We can take η small enough so that

$$| \; Dg(x) \; | \geq \frac{\alpha_f}{2\cdot\hat{K}_f}\cdot|x-1|^{\alpha_g - 1}. \tag{3.27}$$

Note that $\alpha_f - \eta < \alpha_g < \alpha_f + \eta$ imply

$$\frac{\alpha_f - 1 - \eta}{\alpha_f + \eta} < \frac{\alpha_g - 1}{\alpha_g} < \frac{\alpha_f - 1 + \eta}{\alpha_f - \eta}. \tag{3.28}$$

From Chain Rule Theorem, (3.19), (3.27), (3.25), (3.18) and (3.28) yields

$$\begin{aligned} |Dg^l(x)| &= | \; Dg^{l-1}(g(x)) \; | \cdot | \; Dg(x) \; | \\ &\geq m^{l-1}\cdot\frac{\alpha_f}{2\hat{K}_f}\cdot|x-1|^{\alpha_g-1} \\ &= C(f)\cdot\hat{\lambda}^l, \end{aligned} \tag{3.29}$$

where the constant $C(f) = \frac{\alpha_f}{2\cdot\hat{K}_f\cdot m}\cdot\Big(\frac{a}{\hat{K}_f+\eta}\Big)^{\frac{\alpha_f-1-\eta}{\alpha_f+\eta}}$ if $\frac{a}{\hat{K}_f+\eta} \geq 1$ or $C(f) = \frac{\alpha_f}{2\cdot\hat{K}_f\cdot m}\cdot\Big(\frac{a}{\hat{K}_f+\eta}\Big)^{\frac{\alpha_f-1+\eta}{\alpha_f-\eta}}$ if $\frac{a}{\hat{K}_f+\eta} < 1$.

Claim. For all $L \in \mathbb{N}$, $\hat{\delta}$ can be chosen in a such way that for all $x \in [1-\hat{\delta}, 1)$ and for all $g \in \mathscr{V}_\eta$ it have $l = l(g,x) > L$. Indeed, using (3.21) and (3.23) we obtain that,

$$l_g(x) \geq l_g\Big(1-\hat{\delta}\Big) > \frac{ln(a) - ln(g(1-\hat{\delta}))}{ln(M)} = L(\hat{\delta}) \tag{3.30}$$

for all $x \in [1-\hat{\delta}, 1)$ and for all $g \in \mathscr{V}_\eta$.

Note that $L(\hat{\delta}) \to \infty$ as $\hat{\delta} \to 0$. Take $\hat{\delta}$ such that $L(\hat{\delta}) > L$. Therefore, from (3.30) we have that $l > L$ and the proof of claim follows.

Take $L_0 \in \mathbb{N}$ such that $C(f) \cdot \hat{\lambda}^{L_0} > 1$. Take λ_3 with $1 < \lambda_3 < min\{(C(f))^{\frac{1}{L_0}} \cdot \hat{\lambda}, \hat{\lambda}\}$. From claim give above applied to $L = L_0$, the inequality (3.29) give us

$$\begin{aligned} \mid Dg^l(x) \mid &\geq C(f) \cdot \hat{\lambda}^l \\ &= C(f) \cdot \hat{\lambda}^{L_0} \cdot \hat{\lambda}^{l-L_0} \\ &\geq \lambda_3^{L_0} \cdot \lambda_3^{l-L_0} \\ &= {\lambda_3}^l. \end{aligned}$$

For C as in hypothesis of Lemma 3.2.12, take $L = L(f, C)$ such that $C \cdot \lambda_3^L > 1$. Again from the claim give above applied to this choosing L we obtain $l > L$ for appropriate choose of $\hat{\delta}$.

The lemma works with $\mathscr{V}_3 = \mathscr{V}_\eta$ for the chosen η and $\delta_3 = \hat{\delta}$. This ends the proof. ■

The following proposition resemble the quasi-hyperbolicity for Misiurewicz maps and its extension to a C^1-neighborhood (see [42], Theorem 6.3, pag. 261 and Theorem 6.4, pag. 262). We improve this result showing that in our case is possible to choose the C^1-neighborhood non depending on the neighborhood of critical point.

Proposition 3.2.13.. *Let us consider a C^3 map $f \in \mathscr{A}_L$ with negative Schwarzian derivative at $[b, 1)$. Then there are C^1-neighborhood $\mathscr{V}_4 = \mathscr{V}_4(f)$ of f in $\mathscr{A}_L$ and constants $C_4 = C_4(f) > 0$, $\delta_4 = \delta_4(f) > 0$, $\lambda_4 = \lambda_4(f) > 1$ satisfying the following properties: If $k \in \mathbb{N}$, $g \in \mathscr{V}_4$ and $x \in dom(g^k)$ are such that $g^k(x) \in (1 - \delta_4, 1]$, then*

$$|Dg^k(x)| \geq C_4 \cdot \lambda_4^k. \tag{3.31}$$

Moreover, if $x \in (1 - \delta_4, 1)$ then

$$|Dg^k(x)| \geq \lambda_4^k. \tag{3.32}$$

Proof. Fix a C^3 map $f \in \mathscr{A}_L$ with negative Schwarzian derivative at $[b, 1)$. Let us consider $C_2 > 0$ and δ_2 given in Lemma 3.2.11 applied for f.

Take $C_4 = \text{mín}\{1, C_2\}$. Applying Lemma 3.2.12 for f and $C = C_4$ we obtain a C^1-neighborhood $\mathscr{V}_3$, the real numbers δ_3 and λ_3 and an integer L. Choose δ_4 such that

$0 < \delta_4 < \text{mín}\{\delta_2, \delta_3\}$. By the conclusion of Lemma 3.2.11 applied to $\delta = \delta_4$, there are λ_2 and a C^1-neighborhood $\mathscr{V}_2$. Let us consider $\mathscr{V}_4 = \mathscr{V}_2 \cap \mathscr{V}_3$ and choose λ_4 in a such way that $1 < \lambda_4 < \text{mín}\{C_2^{\frac{1}{L}} \cdot \lambda_3, \lambda_2\}$. Note that $C_2^{\frac{1}{L}} \cdot \lambda_3 > 1$ because $C_2 > C = C_4$.

Now we prove that the proposition works with $\mathscr{V}_4$, C_4, δ_4 and λ_4 as chosen above.

Fix $g \in \mathscr{V}_4$, $k \in \mathbb{N}$ and $x \in dom(g^k)$ and such that $g^k(x) \in (1 - \delta_4, 1]$.

We decompose the orbit $\{g^i(x)\}_{i=0}^k$ in several blocks as follows:

$$\{x = x_1, g(x_1), ..., g^{k_1-1}(x_1)\}, \{y_1 = g^{k_1}(x_1), g(y_1), ..., g^{l_1-1}(y_1)\},$$

$$\{x_2 = g^{l_1}(y_1), g(x_2), ..., g^{k_2-1}(x_2)\}, \{y_2 = g^{k_2}(x_2), g(y_2), ..., g^{l_2-1}(y_2)\}, ...,$$
$$\{x_m = g^{l_m-1}(y_{m-1}), g(x_m), ..., g^{k_m}(x_m) = y_m = g^k(x)\},$$

where k_1 is the first integer such that $g^{k_1}(x_1) \in (1-\delta_4, 1)$, $l_1 \geq L$ is given by the conclusion of Lemma 3.2.12 applied to y_1, k_2 is the first integer that $g^{k_2}(x_2) \in (1 - \delta_4, 1)$ and so on.

Notice that $k_1 + l_1 + \cdots + k_{m-1} + l_{m-1} + k_m = k$.

Using the Chain Rule Theorem, (3.11) of Lemma 3.2.11, (3.17) of Lemma 3.2.12, and the definitions of C_4 and λ_4 we obtain

$$\begin{aligned}
|Dg^k(x)| &= |Dg^{k_m}(x_m)| \cdot |Dg^{l_{m-1}}(y_{m-1})| \dots |Dg^{k_2}(x_2)| \\
&\quad \cdot |Dg^{l_1}(y_1)| \cdot |Dg^{k_1}(x_1)| \\
&\geq (C_2 \cdot \lambda_2^{k_m}) \cdot \lambda_3^{l_{m-1}} \dots (C_2 \cdot \lambda_2^{k_2}) \cdot \lambda_3^{l_1} \cdot (C_2 \cdot \lambda_2^{k_1}) \\
&= \lambda_2^{k_1+\cdots+k_m}(C_2 \cdot \lambda_3^{l_1}) \dots (C_2 \cdot \lambda_3^{l_{m-1}}) \cdot C_2 \\
&\geq \lambda_2^{k_1+\cdots+k_m} ... \lambda_4^{l_1} \dots \lambda_4^{l_{m-1}} \cdot C_2 \\
&\geq C_2 \cdot \lambda_4^k \\
&\geq C_4 \cdot \lambda_4^k,
\end{aligned}$$

this proves (3.31) of Proposition 3.2.13.

For finish the proof note that if $x \in (1 - \delta_4, 1)$ then in the decomposition of the orbit $\{g^i(x)\}_{i=0}^k$, k_1 no there exists. Therefore following the proof above we can see that the last C_2 do not appears so we obtain (3.32) of Proposition 3.2.13. The proof follows. ∎

Corollary 3.2.14.. *Let us consider a C^3 map $f \in \mathscr{A}_L$ with negative Schwarzian derivative at $[b,1)$. Then there exists a C^1-neighborhood $\mathscr{V}_5 = \mathscr{V}_5(f)$ of f in $\mathscr{A}_L$ such that each $g \in \mathscr{V}_5$ is hyperbolic far away from the critical point* 1.

Proof. Let us consider a C^3 map $f \in \mathscr{A}_L$ with negative Schwarzian derivative at $[b,1)$. Let δ_4 and C^1-neighborhood $\mathscr{V}_4$ given by Proposition 3.2.13. As we just observed, f is hyperbolic for away 1 because f have negative Schwarzian derivative at $[b,1)$ (see note bellow Definition 3.2.10). Then f is hyperbolic in the maximal f-invariant set in the complement of $(1-\delta_4, 1]$. From the openness of the hyperbolicity we can find a C^1-neighborhood $\tilde{\mathscr{V}_4} = \tilde{\mathscr{V}_4}(f)$ of f in $\mathscr{A}_L$ such that all $g \in \tilde{\mathscr{V}_4}$ is hyperbolic in the maximal g-invariant set in the complement of $(1-\delta_4, 1]$. Define $\mathscr{V}_5 = \mathscr{V}_4 \cap \tilde{\mathscr{V}_4}$. Now take $g \in \mathscr{V}_5$ and $\delta < \delta_4$. Then, for all x in the maximal g-invariant contained in $[0, 1-\delta]$ we have that either $\forall k > 1, g^k(x) \notin (1-\delta_4, 1]$ or for some $k_1 > 1, g^{k_1}(x) \in (1-\delta_4, 1]$. In the first case, taking k big enough we have that $|Dg^k(x)| > 1$. In the other case, by (3.31) of Proposition 3.2.13, $\mid Dg^{k_1}(x) \mid \geq C_4 \cdot \lambda_4^{k_1}$.

Now applying the same argument to $g^{k_1}(x)$, we have two alternatives: there is k big enough such that $\mid Dg^{k_1+k}(x) \mid > 1$ or there is k_2 such that $g^{k_2}(g^{k_1}(x)) \in (1-\delta_4, 1)$ in a such case by (3.32) of Proposition 3.2.13 we have that

$$\mid Dg^{k_2+k_1}(x) \mid \geq C_4 \cdot \lambda_4^{k_1} \cdot \lambda_4^{k_2}.$$

Inductively, we obtain that for some k big enough $\mid Dg^k(x) \mid > 1$.

From this and Proposition 3.2.9 we obtain that g is hyperbolic in the maximal g-invariant set in the complement of $(1-\delta, 1]$. As δ is arbitrary then g is hyperbolic far away from the critical point 1. The proof follows. ■

3.2.3. Proof of Theorem 3.2.4

Let $f \in \mathscr{A}_L$ and x a element in the invariant maximal set contained in $[0,a] \cup [b,1]$. The ω-limit set of x, $\omega_f(x)$, is defined by

$$\omega_f(x) = \{y \in [0,a] \cup [b,1] : \exists\ n_i \to \infty \text{ such that } f^{n_i}(x) \to y\}.$$

Let J be an interval contained in maximal invariant set in $[0,a] \cup [b,1]$. The ω-limit set of J, $\omega_f(J)$, is the union of the $\omega_f(x)$ for $x \in J$. We said that J is a *wandering interval* of the map f if the intervals J, $f(J), \ldots$ are pairwise disjoint and the ω-limit set of J is not equal to a single periodic orbit.

The Theorem 3.2.4 is related with non-existence of wandering intervals. This old problem goes back to Poincaré's work dealing with homeomorphisms of the circle (see [58]). Since then efforts of a number of authors have been directed towards proving the non-existence of wandering intervals because their appearance complicates our understanding of the dynamics.

Proof of Theorem 3.2.4. Fix a C^3 map $f \in \mathscr{A}_L$ with negative Schwarzian derivative at $[b, 1)$. Let us consider $\mathscr{V}_4$, δ_4 y λ_4 given by Proposition 3.2.13. Take a C^1-neighborhood $\mathscr{V}_5$ of f in $\mathscr{A}_L$ given by Corollary 3.2.14. Define $\mathscr{V} = \mathscr{V}_4 \cap \mathscr{V}_5$.

Now fix $g \in \mathscr{V}$.

First we observe that g has no sinks. Indeed, by Corollary 3.2.14, g is hyperbolic (expanding) far away from critical point 1 and by Proposition 3.2.13 applied to orbits beginning close to de critical point 1 we conclude that g has no sinks.

Suppose that $\Lambda_g = \bigcap_{i=0}^{\infty} g^{-i}(I)$ contains an interval J. If there are integers $m \neq n$ such that $g^m(J) \cap g^n(J)$ has non empty interior, then g has sinks (see [G 79], Lemma A, pag. 142) and so we get a contradiction. Therefore, the sequence of intervals $\{g^n(J)\}_{n=0}^{\infty}$ are pairwise disjoint and can not accumulate a sink, i.e. J is a wandering interval. From Corollary 3.2.14 it follows that $g^n(J)$ accumulate to 1 and

$$| g^n(J) | \to 0 \text{ as } n \to +\infty. \tag{3.33}$$

Let us consider $0 < \eta < \delta_4$ and an integer n_0 in a such way that $\forall n \geq n_0$, $|g^n(J)| < \delta_4 - \eta$. So, if for $n \geq n_0$ $g^n(J), \cap (1-\eta, 1) \neq \emptyset$ then $g^n(J) \subset (1-\delta_4, 1)$.

As the iterates of J accumulates 1, there is a sequence n_k such that $g^{n_k}(J) \subset (1-\delta_4, 1)$. From Chain Rule Theorem we have

$$\begin{aligned} | g^{n_k}(J) | &= | g^{n_k - n_0}(g^{n_0}(J)) | \\ &= | Dg^{n_k - n_0}(\xi) | \cdot | g^{n_0}(J) |, \end{aligned}$$

for some $\xi \in g^{n_0}(J) \subset (1-\delta_4, 1)$.

Because $g^{n_k-n_0}(\xi) \in g^{n_k}(J) \subset [1-\delta_4, 1]$ we can apply (3.32) to obtain

$$\mid g^{n_k}(J) \mid \ \geq \lambda_2^{n_k-n_0} \cdot \mid g^{n_0}(J) \mid .$$

As $n_k \to \infty$ we have that

$$\mid g^{n_k}(J) \mid \to \infty$$

and so we get a contradiction with (3.33). The proof follows. ■

3.3. Proof of the Main Theorem

In this section we prove the Main Theorem (Theorem Theorem 3.1.1)

Let $X \in \mathscr{X}^\infty(M, \partial M)$ be exhibiting a generic singular cycle

$$\Gamma = \{\sigma, O, \gamma_0, \gamma_1\}$$

associated to a singularity $\sigma \in \partial M$. We shall prove that X also exhibits, for all k large enough, a C^k-robust transitive set Λ on the space of C^∞ vector fields containing σ.

Under generic hypothesis we can assume the existence of C^2 linearizing coordinates in a neighborhood of the singularity σ as well for the Poincaré map associated to the periodic orbit O (see [80], [27]). Moreover, there is a k, depending only on the eigenvalues of σ and O, such that for any C^k vector field Y C^k-close to X, there are C^2 linearizing coordinates in neighborhoods of the continuation of the singularity σ as well for the Poincaré map associated to the continuation of the periodic orbit O, which depend smoothly on Y. From now on we fix such k.

We split the proof in some steps. In the first we describe how the cycle is located. The second prove the existence of a isolating block and finally, in the third step we show the transitivity.

First step: To describe how the cycle Γ is located in ∂M. As $\sigma \in \partial M$ we have either $W^s(\sigma) \subset \partial M$ or $W^u(\sigma) \subset \partial M$ (this follows from dominated splitting arguments applied to the eigenvector decomposition to $T_\sigma(\partial M)$, see [3]). In any case we have $O \subset \partial M$ because of the existence of γ_0, γ_1 and the fact that ∂M is compact. If $W^s(\sigma) \subset \partial M$ then $W^u(O) \not\subset \partial M$ otherwise these manifolds could not be transverse at γ_1, therefore $W^s(O) \subset \partial M$. Hence $\gamma_0 \subset \partial M$ and so $W^u(\sigma) \subset \partial M$ at σ which is a contradiction. This proves that $W^u(\sigma) \subset \partial M$. From this it follows that $W^s(\sigma)$ is transverse to ∂M at σ and $W^s(O) \subset \partial M$. Therefore $W^{ss}(\sigma) \subset \partial M$ because $W^s(O)$ is transversal to $W^{cu}(\sigma)$ along γ_0. All of this proves that the cycle is like that in Figure 3.5.

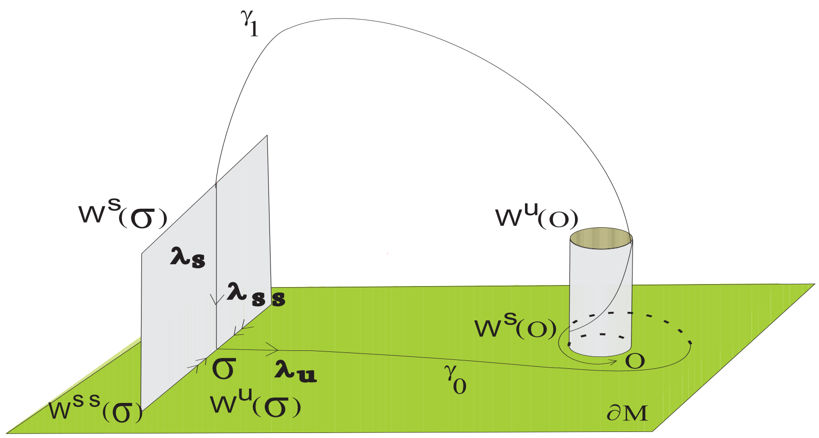

Figura 3.5: Cycle Γ located in ∂M.

Second step: To construct a suitable neighborhood U of Γ. Fix a point $p \in O$ and a cross section S at p. Under generic hypothesis we can assume the existence of $\psi_X : S \to \Sigma$ a C^3 change coordinate on S that linearize the Poincaré map defined on S, where

$$\Sigma = [-\delta, 1+\delta] \times [0, 1+\delta],$$

with $\delta > 0$ small.

In this coordinates we have $p = (0,0)$, $\Sigma \cap \partial M = \partial\Sigma \cap \partial M = \{y = 0\} \subset W^s(O)$ and $\{x = 0\} \subset W^u(O)$.

Identifying S with Σ trough ψ_X, we can select one point $q_1 \in \gamma_1 \cap \Sigma$ whose positive orbit does not meet Σ. Without loss of generality we can assume $q_1 = (0,1)$. Then the connected component of $W^s(\sigma) \cap \Sigma$ that contains $(1,0)$ is the graph of a smooth map φ. Also we can assume that the Poincaré map is defined on $[-\delta, 1+\delta] \times [0, a]$ for some a fix with $0 < a < 1$.

We may think of Σ as the meridian disk of a solid torus neighborhood

$$ST = \Sigma \times S^1$$

of O in M where S^1 denotes the unit circle.

Afterward we select one point $q_0 \in \gamma_0 \cap \Sigma$ whose negative orbit does not meet Σ. Without loss of generality we can assume $q_0 = (1, 0)$.

Take linearizing coordinates on a neighborhood of singularity σ. In this coordinates $\sigma = (0, 0, 0)$ and $B_\sigma = \{(x_1, x_2, x_3) : \ 0 \leq x_3 < 1, \, -1 < x_1, x_2 < -1\}$ is a neighborhood of σ. Moreover, $\{x_3 = 0\} \subset \partial M$ so that $\{x_2 = 0\} \subset W^s(\sigma)$, $\{x_1 = 0\} \subset W^{cu}(\sigma)$, $\{x_1 = x_2 = 0\} \subset W^{ss}(\sigma)$ and $\{x_2 = x_3 = 0\} \subset W^u(\sigma)$. We can also assume that the positive orbits through $\{y = \varphi(x)\}$ go directly to $\{x_3 = 1\}$ without intersect Σ. Hence the same happens in some horizontal band $\hat{H}_1$ around $\{y = \varphi(x)\}$. By taking the union of the positive orbits through this band we obtain a compact set denoted by W_1.

Finally, $(0, 1, 0)$ (in the coordinate system (x_1, x_2, x_3)) belong to γ_0. Hence we can choose a small tubular neighborhood W_0 around the orbit path in γ_0 joining $(0, 1, 0)$ to $q_0 = (1, 0)$.

Collecting the sets ST, W_1, B_σ and W_0 together we obtain

$$U = ST \cup W_1 \cup B_\sigma \cup W_0$$

which is a neighborhood of Γ.

The principal facts given above are as in the figure 3.6.

Third step (the most important)**:** To prove that the set

$$\Lambda = \bigcap_{t \in \mathbb{R}} X_t(U)$$

is a C^k-robust transitive set of X. Indeed, in this work we will prove that for all Y C^k-close to X, $\Lambda_Y = \bigcap_{t \in \mathbb{R}} Y_t(U)$ is a transitive set. Note that Theorem 3.1.1 follows from from this step since $\sigma \in \Gamma \subset \Lambda$.

To prove that Λ is a C^k-robust transitive set for X we need to do an accurate description for Poincaré maps involved in the dynamic around the cycle. With this we will establish results about hyperbolicity for orbits far away from stable manifold of the singularity.

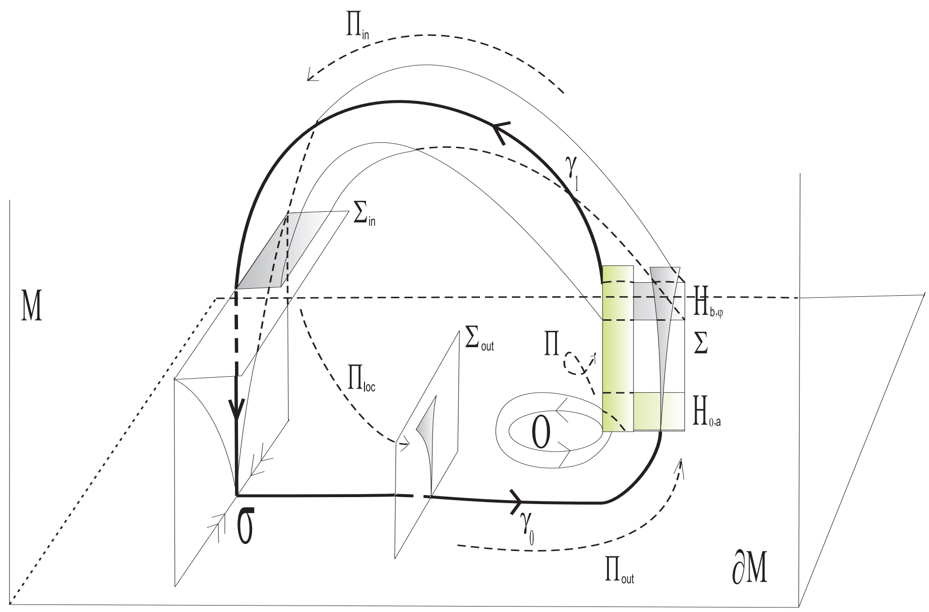

Figura 3.6: The Poincaré map on the cross-section on O.

First of all we observe that Λ is not empty since it contains the cycle Γ. On the other hand, by choosing carefully the boundaries of the sets which compose the isolating block U(given in the second step), we can observe that U is a compact neighborhood of Λ. Hence Λ is a non-trivial isolated subset of X.

Let consider the Poincaré map Π_X associated to the periodic orbit O. By linearizing hypothesis we have $\Pi_X : H_{0,a} \subset \Sigma \to \Sigma$ is the linear map

$$(x, y) \to (\lambda \cdot x, \rho \cdot y); \tag{3.34}$$

where $0 < \lambda < 1$ and $\rho > 1$ are the eigenvalues for the periodic orbit O, $H_{0,a}$ is the horizontal band $[0, 1+\delta] \times [0, a]$; and $\Pi_X(\{y = a\}) \subset \{y = 1+\delta\}$.

Let $\Sigma_{in} = \{x_3 = 1, x_2 > 0\}$ and $\Sigma_{out} = \{x_2 = 1\}$.

The flow define a Poincaré map $\Pi_{in,X} : \hat{H}_1 \subset \Sigma \to \Sigma_{in}$ defined in a horizontal band $\hat{H}_1$ around $\{y = \varphi(x)\}$. We have that one of the components of $\hat{H}_1 \setminus \{(x, \varphi(x)) : 0 \leq x \leq 1\}$ goes to $\{x_3 = 1, x_2 < 0\}$ while the another one goes to $\{x_3 = 1, x_2 > 0\}$. We denote by H_1 the closure of the component which arrives to $\{x_3 = 1, x_2 > 0\}$. We may assume that H_1 is bounded by the curves $\{y = \varphi_1(x)\}$ and $\{y = \varphi(x)\}$. Depending on the orientation of

$\Pi_{in,X}$ at $(0,1)$ we have for every x either $\varphi_1(x) < \varphi(x)$ or $\varphi_1(x) > \varphi(x)$. We assume that $\varphi_1(x) < \varphi(x)$, the other case can be handle with similar arguments. Furthermore changing the section Σ_{in} we can assume that $\varphi_1(x) = b$, for some b fix with $0 < b < 1$. We denote $H_1 = H_{b,\varphi} = \{(x,y) : x \in [0, 1+\delta],\ b \le y \le \varphi(x)\}$.

Denote by $\alpha_X = -\frac{\lambda_s}{\lambda_u}$ and $\beta_X = -\frac{\lambda_{ss}}{\lambda_u}$.

There is also a (local) Poincaré map $\Pi_{loc,X} : \Sigma_{in} \to \Sigma_{out}$. By linearizing hypothesis $\Pi_{loc,X}(x_1, x_2) = (x_1 \cdot x_2^{\beta_X}, x_2^{\alpha_X})$. We can extend continuously this map to $\{x_3 = 1, x_2 \ge 0\}$ by setting $\Pi_{loc,X}(\{x_2 = 0\}) = (0, 1, 0)$.

There is third a Poincaré map $\Pi_{out,X}$ from $W_0 \cap \{x_2 = 1\}$ (which is a neighborhood of $(0,1,0)$ in $\{x_2 = 1\}$) into $W_0 \cap \Sigma$ (which is a neighborhood of q_0 in Σ).

Putting these Poincaré maps together we obtain a return map

$$R_X : H_{0,a} \cup H_{b,\varphi} \subset \Sigma \to \Sigma$$

defined as Π_X in $H_{0,a}$ and $\Pi_{out,X} \circ \Pi_{loc,X} \circ \Pi_{in,X}$ in H_1. We write $\Pi_X = \Pi, \Pi_{in,X} = \Pi_{in}$, etc. to emphasize dependence on X. We can suppose that $R_X(\{y = b\}) \subset [0, 1+\delta] \times (1, 1+\delta)$.

Let $O(Y)$, $\sigma(Y)$, φ_Y, $q_0(Y)$, $q_1(Y)$, $\gamma_0(Y)$ and $\gamma_1(Y)$ be the continuations of O, σ, φ, q_0, q_1, γ_0 and γ_1 respectively. Observe that the cross-section S remain transverse to any flow C^1-close to X. Take $\psi_Y : S \to \Sigma$ a C^2 change coordinate on S that linearize the Poincaré map induced by the Y-flow on S. Identifying S with Σ trough ψ_Y we can suppose that $q_0(Y) = (1,0)$ and $q_1(Y) = (0,1)$. Moreover let λ_Y and ρ_Y the eigenvalues associated to $O(Y)$, the continuation of λ and ρ respectively. Let $\lambda_{ss}(Y) < \lambda_s(Y) < 0 < \lambda_u(Y)$ be the eigenvalues of $DY(\sigma(Y))$, and denote $\alpha_Y = -\frac{\lambda_s(Y)}{\lambda_u(Y)}$ and $\beta_Y = -\frac{\lambda_{ss}(Y)}{\lambda_u(Y)}$.

The maps $\Pi_X, \Pi_{in,X}, \Pi_{loc,X}, \Pi_{out,X}$ as well as R_X have continuations $\Pi_Y, \Pi_{in,Y}$, $\Pi_{loc,Y}$, $\Pi_{out,Y}$ and R_Y for all vector field Y close to X. We can suppose that $R_Y(\{y = b\}) \subset [0, 1+\delta] \times (1, 1+\delta)$.

We have that

$$\bigcap_{t\in\mathbb{R}} Y_t(U) = Cl\left(\bigcup_{t\in\mathbb{R}} Y_t\left(\bigcap_{n\in\mathbb{Z}} R_Y^n(H_{0,a} \cup H_{b,\varphi_Y})\right)\right),$$

where $Cl(\cdot)$ denotes the closure operator. So, in order to prove that $\cap_{t\in\mathbb{R}} Y_t(U)$ is a transitive set, we only need to prove that the maximal invariant set

$$\bigcap_{n\in\mathbb{Z}} R_Y^n(H_{0,a}\cup H_{b,\varphi_Y}) \tag{3.35}$$

is a transitive set of R_Y.

Previously, we recall that a continuous foliation $\mathscr{F}$ is *invariant* by $R : Dom(R) \to \mathbb{R}^2$ if for every leaf L of $\mathscr{F}$ contained in $Dom(R)$ there is a leaf $\tilde{L}$ of $\mathscr{F}$ such that $R(L) \subset \tilde{L}$. We say that $\mathscr{F}$ is *contracting* by R is there exists a constant $C > 0$ and $0 < \lambda < 1$ such that

$$\| DR^n(p)\cdot v \| \leq C\cdot\lambda^n,$$

for all $n \in \mathbb{N}$, for all $p \in L$, for all $L \in \mathscr{F}$ and for all $v \in T_pL$.

For all vector field Y C^k-close to X, there exists a contracting invariant C^1 foliation $\mathscr{F}$ for R_Y depending smoothly on Y(see [6]). Using that $R_Y(\{y = b\}) \subset [0, 1+\delta]\times(1, 1+\delta)$, this construction can be made in a such way that the set $\{y = 0\}$, $\{y = a\}$, $\{y = b\}$ and $\{y = \varphi_Y(x)\}$ are leaves of this foliation. Moreover, by condition on the eigenvalues of the singularity σ, the invariant foliation for the vector field X can be choosing C^3 (see [71]). We can use this foliation to put new coordinates (x, y) on Σ, still linearizing, such that for all $(x, y) \in H_{0,a}\cup H_{b,1}$

$$R_Y(x, y) = (f_Y(x, y), g_Y(y)), \tag{3.36}$$

for some C^1 maps $f_Y(\cdot,\cdot)$ and $g_Y(\cdot)$. Moreover $f_X(\cdot,\cdot)$ and $g_X(\cdot)$ are C^3 maps. Note that $g_Y(a)$ and $g_Y(b)$ are greater than 1.

Proposition 3.3.1. *Let $X \in \mathscr{X}^\infty(M,\partial M)$ and k as above. Then for all Y C^k-close to X, the map $g_Y(\cdot)$ satisfies the following properties:*

(a) $g_Y(\cdot) \in \mathscr{A}_L$;

(b) $g_Y(\cdot)$ is C^1 close to $g_X(\cdot)$;

(c) $g_X(\cdot)$ is a C^3 map and $Sg_X(y) < 0$ at $y \in [b, 1)$.

Proof. First of all we will obtain the expression of $g_Y(y)$.

From (5.2) and (3.34) we get that

$$g_Y(y) = \rho_Y \cdot y \tag{3.37}$$

for all $y \in [0, a]$.

Now we will obtain the expression of $g_Y(y)$ for all $y \in [b, 1]$.

Indeed, by Taylor formula,

$$\Pi_{in,Y}(x, y) = \Pi_{in,Y}(q_1(Y)) + D\Pi_{in,Y}(q_1(Y)) \cdot (x, y-1) + \Theta_{in,Y}(x, y-1),$$

where

$$D\Pi_{in,Y}(q_1(Y)) = \begin{bmatrix} a_Y & b_Y \\ c_Y & d_Y \end{bmatrix}$$

with $a_Y, b_Y, c_Y, d_Y \in \mathbb{R}$ and $\text{lím}_{(x,y)\to(0,1)} \frac{\Theta_{in,Y}(x,y-1)}{\|(x,y-1)\|} = 0$.

As $D\Pi_{in,Y}(q_1(Y)) \cdot e_1 = (a_Y, c_Y)$ and $D\Pi_{in,Y}(q_1(Y)) \cdot e_2 = (b_Y, d_Y)$, by assumption we have that a_Y and d_Y are negatives and $c_Y = 0$. Also we have that $\Pi_{in,Y}(q_1(Y)) = (x_0(Y), 0)$. Then

$$\begin{aligned} \Pi_{in,Y}(x, y) &= (x_0(Y) + a_Y \cdot x + b_Y \cdot (y-1) + \Theta_{1,Y}(x, y-1), \\ &\quad d_Y \cdot (y-1) + \Theta_{2,Y}(x, y-1)), \end{aligned}$$

where $\Theta_{in,Y} = (\Theta_{1,Y}, \Theta_{2,Y})$.

Note that $\Theta_{2,Y}(x, 0) = 0$.

Also, by Taylor formula,

$$\Pi_{out,Y}(\tilde{x}_1, \tilde{x}_3) = \Pi_{out,Y}(0, 0) + D\Pi_{out,Y}(0, 0) \cdot (\tilde{x}_1, \tilde{x}_3) + \Theta_{out,Y}(\tilde{x}_1, \tilde{x}_3),$$

where

$$D\Pi_{out,Y}(0, 0) = \begin{bmatrix} \tilde{a}_Y & \tilde{b}_Y \\ \tilde{c}_Y & \tilde{d}_Y \end{bmatrix}$$

with $\tilde{a}_Y, \tilde{b}_Y, \tilde{c}_Y, \tilde{d}_Y \in \mathbb{R}$ and $\text{lím}_{(\tilde{x}_1,\tilde{x}_3)\to(0,0)} \frac{\Theta_{out,Y}(\tilde{x}_1,\tilde{x}_3)}{\|(\tilde{x}_1,\tilde{x}_3)\|} = 0$.

As $D\Pi_{out,Y}(0, 0) \cdot e_1 = (\tilde{a}_Y, \tilde{c}_Y)$ and $D\Pi_{out,Y}(0, 0) \cdot e_2 = (\tilde{b}_Y, \tilde{d}_Y)$, by assumption we have that $\tilde{a}_Y$ and $\tilde{d}_Y$ are positives and $\tilde{c}_Y = 0$. Moreover we supposed that $\Pi_{out,Y}(0, 0) = (1, 0)$.

Then

$$\begin{aligned}\Pi_{out,Y}(\tilde{x}_1,\tilde{x}_3) &= (1+\tilde{a}_Y\cdot\tilde{x}_1+\tilde{b}_Y\cdot\tilde{x}_3+\Theta_{3,Y}(\tilde{x}_1,\tilde{x}_3),\\ &\quad \tilde{d}_Y\cdot\tilde{x}_3+\Theta_{4,Y}(\tilde{x}_1,\tilde{x}_3)),\end{aligned}$$

where $\Theta_{out,Y}=(\Theta_{3,Y},\Theta_{4,Y})$.

Note that $\Theta_{4,Y}(\tilde{x}_1,0)=0$.

On the other hand, remember that

$$\Pi_{loc,Y}(x_1,x_2)=(x_1\cdot x_2^{\beta_Y},x_2^{\alpha_Y}).$$

Putting

$$A(Y)=\Big[x_0(Y)+a_Y\cdot x+b_Y\cdot(y-1)+\Theta_{1,Y}(x,y-1)\Big]$$

and

$$B(Y)=\Big[d_Y\cdot(y-1)+\Theta_{2,Y}(x,y-1)\Big]^{\beta_Y},$$

we denote by

$$\tilde{x}_1=A(Y)\cdot B(Y)$$

and

$$\tilde{x}_3=\Big[d_Y\cdot(y-1)+\Theta_{2,Y}(x,y-1)\Big]^{\alpha_Y}.$$

As $R_Y=\Pi_{out,Y}\circ\Pi_{loc,Y}\circ\Pi_{in,Y}$ (in $H_{b,1}$) from (5.2) we get that

$$f_Y(x,y)=1+\tilde{a}_Y\cdot\tilde{x}_1+\tilde{b}_Y\cdot\tilde{x}_3+\Theta_{3,Y}(\tilde{x}_1,\tilde{x}_3)$$

and

$$\begin{aligned}g_Y(y) &= \tilde{d}_Y\cdot\Big[d_Y\cdot(y-1)+\Theta_{2,Y}(x,y-1)\Big]^{\alpha_Y}+\\ &\quad \Theta_{4,Y}(\tilde{x}_1,\tilde{x}_3).\end{aligned}\tag{3.38}$$

Therefore, $g_Y(y)$ can to express as

$$g_Y(y)=|y-1|^{\alpha_Y}\cdot H_Y(y),\tag{3.39}$$

where

$$H_Y(y) = \tilde{d}_Y \cdot \Big[- d_Y + \frac{\Theta_{2,Y}(x, y-1)}{|y-1|} \Big]^{\alpha_Y} + \frac{\Theta_{4,Y}(\tilde{x}_1, \tilde{x}_3)}{|y-1|^{\alpha_Y}}. \tag{3.40}$$

Next, by expressions for $\tilde{x}_1$ and $\tilde{x}_3$, we have that

$$\lim_{y \to 1} H_Y(y) = \tilde{d}_Y \cdot |d_Y|^{\alpha_Y} > 0. \tag{3.41}$$

Let us prove Statement (a). By (3.37), $g_Y(y) = \rho_Y \cdot y$ for all $y \in [0, a]$. Moreover, by assumption we have that g_Y is decreasing on $[b, 1]$, $g_Y(1) = 0$, $g_Y(a) > 1$ and $g_Y(b) > 1$.

To continue we need compute the derivative. From (3.39) we have

$$Dg_Y(y) = |y-1|^{\alpha_Y - 1} \cdot \Big[- \alpha_Y \cdot H_Y(y) + |y-1| \cdot DH_Y(y) \Big]. \tag{3.42}$$

On the other hand, from (3.38) we have

$$\begin{aligned} Dg_Y(y) \;=\; & |y-1|^{\alpha_Y - 1} \cdot \Big(\alpha_Y \tilde{d}_Y \cdot \Big(- d_Y + \frac{\Theta_{2,Y}(x, y-1)}{|y-1|} \Big)^{\alpha_Y - 1} \cdot \\ & (d_Y + \partial_y \Theta_{2,Y}(x, y-1)) + \partial_{\tilde{x}_1} \Theta_{4,Y}(\tilde{x}_1, \tilde{x}_3) \cdot \frac{\partial_y \tilde{x}_1}{|y-1|^{\alpha_Y - 1}} + \\ & + \partial_{\tilde{x}_3} \Theta_{4,Y}(\tilde{x}_1, \tilde{x}_3) \cdot \frac{\partial_y \tilde{x}_3}{|y-1|^{\alpha_Y - 1}} \Big). \end{aligned} \tag{3.43}$$

Moreover note that

$$\begin{aligned} \frac{\partial_y \tilde{x}_1}{|y-1|^{\alpha_Y - 1}} \;=\; & |y-1|^{\beta_Y - \alpha_Y + 1} \cdot (b_Y + \partial_y \Theta_{1,Y}(x, y-1)) \cdot \\ & \Big(- d_Y + \frac{\Theta_{2,Y}(x, y-1)}{|y-1|} \Big)^{\beta_Y} + \\ & \beta_Y \cdot |y-1|^{\beta_Y - \alpha_Y} \cdot (d_Y + \partial_y \Theta_{2,Y}(x, y-1)) \\ & \cdot (x_0(Y) + a_Y \cdot x + b_Y \cdot (y-1) + \Theta_{1,Y}(x, y-1)) \\ & \cdot \Big(- d_Y + \frac{\Theta_{2,Y}(x, y-1)}{|y-1|} \Big)^{\beta_Y - 1}. \end{aligned}$$

From this we get that $\lim_{y \to 1} \frac{\partial_y \tilde{x}_1}{|y-1|^{\alpha_Y - 1}} = 0$.

Furthermore,

$$\begin{aligned} \frac{\partial_y \tilde{x}_3}{|y-1|^{\alpha_Y - 1}} \;=\; & \Big(- d_Y + \frac{\Theta_{2,Y}(x, y-1)}{|y-1|} \Big)^{\alpha_Y - 1} \\ & \cdot (d_Y + \partial_y \Theta_{2,Y}(x, y-1)) \end{aligned}$$

From this we get that $\lim_{y\to 1} \frac{\partial_y \tilde{x}_3}{|y-1|^{\alpha_Y - 1}} = -|d_Y|^{\alpha_Y}$.

Comparing (3.42) with (3.43) we get

$$\lim_{y\to 1} \Big(- \alpha_Y \cdot H_Y(x,y) + |y-1| \cdot DH_Y(y) \Big) = -\alpha_Y \cdot \tilde{d}_Y \cdot |d_Y|^{\alpha_Y}.$$

So, from this and (3.41) we obtain

$$\lim_{y\to 1} |y-1| \cdot DH_Y(y) = 0.$$

Therefore, $Dg_Y(1) = 0$. All of this proved that (i) and (ii) in definition 3.2.3. This completes the proof of (a).

Let us prove Statement (b). First of all note that α_Y is close to α_X because depend on the eigenvalues of the singularity σ, which move continuously with the vector fields. On the other hand, the above argument show that the values for the entries of $D\Pi_{in,Y}(q_1)$ and $D\Pi_{out,Y}(0,0)$ are close. Then, to prove that $H_Y(\cdot)$ is C^0 close to $H_X(\cdot)$ only remain to show that the quotients involving the remainder term in (3.40) for X and Y respectively are close. We made it using Taylor's Formula for C^2 maps. In the same way we conclude that $Dg_Y(\cdot)$ in 3.42 is C^0 close to $Dg_X(\cdot)$. This proves Statements (b).

Let us prove Statement (c). Indeed,

$$\begin{aligned} Dg_X(y) &= -\alpha_X \cdot \tilde{d}_X \cdot |d_X|^{\alpha_X} \cdot |y-1|^{\alpha_X - 1} \cdot \tilde{A}(y) \\ D^2 g_X(y) &= \alpha_X \cdot (\alpha_Y - 1) \cdot \tilde{d}_X \cdot |d_X|^{\alpha_X} \cdot |y-1|^{\alpha_X - 2} \cdot \tilde{B}(y) \end{aligned}$$

and

$$D^3 g_X(y) = -\alpha_X \cdot (\alpha_X - 1) \cdot (\alpha_X - 2) \cdot \tilde{d}_X \cdot |d_X|^{\alpha_X} \cdot |y-1|^{\alpha_X - 3} \cdot \tilde{C}(y),$$

where $\tilde{A}(y), \tilde{B}(y)$ and $\tilde{C}(y)$ are maps such that $\lim_{y\to 1} \tilde{A}(y) = 1$, $\lim_{y\to 1} \tilde{B}(y) = 1$ and $\lim_{y\to 1} \tilde{C}(y) = 1$.

Therefore, for all y we have,

$$\begin{aligned} Sg_X(y) &= \frac{D^3 g_X(y)}{Dg_X(y)} - \frac{3}{2} \cdot \Big(\frac{D^2 g_X(y)}{Dg_X(y)} \Big)^2 \\ &= |y-1|^{-2} \cdot h_X(y), \end{aligned}$$

where

$$h_X(y) = \frac{2 \cdot (\alpha_X - 1) \cdot (\alpha_X - 2) \cdot \tilde{A}(y) \cdot \tilde{C}(y) - 3 \cdot (\alpha_X - 1)^2 \cdot (\tilde{B}(y))^2}{2 \cdot (\tilde{A}(y))^2}.$$

Then, taken limit we obtain that $h_X(1) = 1 - \alpha_X^2 < 0$.

Replacing q_0 by some positive iterate of it, say n and replacing q_1 by some negative iterate of it, say the same $-n$, we obtain a new Poincaré map $\hat{R}$ defined on a small neighborhood of $(0,0)$ on $\hat{\Sigma}$. Normalizing by the change coordinates $(\hat{x}, \hat{y}) = (\lambda^{-n}x, \rho^n y)$, the shape of $\hat{R} = \Pi^n \circ \Pi_{out} \circ \Pi_{loc} \circ \Pi_{in} \circ \Pi^n$ in this new coordinates is given by:

$$\hat{R}(\hat{x}, \hat{y}) = (f(\lambda^{2n}\hat{x}, \hat{y}), \rho^{2n} g(\hat{y})).$$

This Poincaré map $\hat{R} = (\hat{f}, \hat{g})$ is defined on $[0,1] \times [\hat{b}, 1]$ for some $\hat{b}$ very close to 1. Take this in mind it is clear that the map $\hat{g}$ has negative Schwarzian derivative. So item c) is proved.

Therefore choosing a small horizonal band around $\{y = 1\}$, modifying suitably the section $\hat{\Sigma}$ and iterating by the Poincaré map around the orbit O, we can suppose that $Sg_X(y) < 0$ for all $y \in [b, 1)$. This proves the Statements (c) and finish the proof of Proposition. ■

Denote by $T\Sigma$ the tangent bundle of Σ. Given $p \in \Sigma$, $\gamma > 0$, we denote by $C_H^\gamma(p)$ the horizontal γ-cone with inclination γ, i.e.,

$$C_H^\gamma(p) = \{v \in T_p\Sigma : v = (u, w); |w| \leq \gamma \cdot |u|\}.$$

Also, we denote by $C_V^\gamma(p)$ the vertical γ-cone with inclination γ, i.e.,

$$C_V^\gamma(p) = \{v \in T_p\Sigma : v = (u, w); |u| \leq \gamma \cdot |w|\}.$$

A γ-cone field in Σ is a continuous map $C^\gamma : p \in \Sigma \mapsto C^\gamma(p) \subset T_p\Sigma$, where $C^\gamma(p)$ is a γ-cone with constant inclination γ on $T_p\Sigma$. Let $R : H_{0,a} \cup H_{b,1} \subset \Sigma \to \Sigma$ be any map. A γ-cone field C^γ is called *R-invariant* if $DR(C^\gamma(p) \setminus \{0\}) \subset int(C^\gamma(R(p)))$ for all $p \in H_{0,a} \cup H_{b,1}$. A γ-cone field C^γ is called *R-contracting* if there are $C > 0$ and $0 < \lambda < 1$ such that $\| DR^n(p) \cdot v \| \leq C \cdot \lambda^n \cdot \| v \|$, $\forall n \in \mathbb{N}$, $\forall p \in \Sigma$ and $\forall v \in C^\gamma(p)$. A γ-cone field C^γ is called *transversal* to a foliation $\mathscr{F}$ on Σ if $T_pL \cap C^\gamma(p) = \{0\}$, $\forall p \in L$ and $\forall L \in \mathscr{F}$.

Proposition 3.3.2.. *Let $X \in \mathscr{X}^\infty(M, \partial M)$ be exhibiting a generic singular cycle as in Theorem 3.1.1. Then exists γ with $0 < \gamma \leq 1$ such that for all Y C^1-close to X there are invariants γ-cone fields C_H^γ and C_V^γ on Σ. Moreover C_H^γ is R_Y^{-1}-contracting and C_V^γ is transversal to the foliation $\mathscr{F}$.*

Proof. Fix $X \in \mathscr{X}^\infty(M, \partial M)$ as in the Statements of Proposition. Then there exists a γ with $0 < \gamma \leq 1$ such that for all Y C^1-close to X there is a horizontal γ-cone field C_H^γ on Σ which is invariant and contracting by R_Y^{-1} (see [6]).

On the other hand, using that Λ is a partial hyperbolic set for X, for some $\gamma \in (0, 1]$ we can construct in standard way an R_X-invariant vertical γ-cone field C_V^γ on an open set U containing Λ (see [8], Lemma 4, pag. 284-285 or also [52]). Then, from the openness we have that for all vector field Y close to X this cone field is R_Y-invariant. Moreover this cone field is transversal to the foliation $\mathscr{F}$. Therefore, the proof follows. ■

From now on we fix such γ, C_H^γ and C_V^γ.

Proposition 3.3.3.. *Let $X \in \mathscr{X}^\infty(M, \partial M)$ be exhibiting a generic singular cycle as in Theorem 3.1.1. Then, there exist a neighborhood $\mathscr{V}$ of X in $\mathscr{X}^k(M, \partial M)$ such that: For all $\delta > 0$, for all $Y \in \mathscr{V}$, R_Y is hyperbolic on $([0,1]\times[0,a])\cup([0,1]\times[b,1-\delta])$, More precisely, there are $C_1 = C_1(Y) > 0$, $C_2 = C_2(R_Y, \delta) > 0$, $0 < \lambda_1 = \lambda_1(Y) < 1$, $\lambda_2 = \lambda_2(R_Y, \delta) > 1$ in a such way that $\forall n \in \mathbb{N}$, $\forall p$ with $R^i(p) \in [0,1] \times ([0,a] \cup [b, 1-\delta])$, $0 \leq i \leq n-1$ and $\forall v \in C_H^\gamma(p)$ we have*

$$\| DR_Y^{-n}(p) \cdot v \| < C_1 \cdot \lambda_1^n \cdot \| v \| . \tag{3.44}$$

Moreover, for all $v \in C_V^\gamma(p)$ we have

$$\| DR_Y^{n}(p) \cdot v \| > C_2 \cdot \lambda_2^n \cdot \| v \| . \tag{3.45}$$

Proof. Fix $X \in \mathscr{X}^\infty(M, \partial M)$ as in the statement of Proposition.

By Proposition 3.3.2 we can choice a neighborhood $\mathscr{V}_1$ of X in $\mathscr{X}^1(M, \partial M)$ such that for all $Y \in \mathscr{V}_1$ the γ-cone field C_H^γ and C_V^γ exist. Moreover, C_H^γ is invariant and contracting by R_Y^{-1} and C_V^γ is invariant by R_Y and transversal to the foliation $\mathscr{F}$.

Note that X has associate an one dimensional map g_X. From Proposition 3.3.1 (see statements (c)) we get that g_X is a C^3 map, $g_X \in \mathscr{A}_L$ and g_X has negative Schwarzian derivative at [b,1). Then by Corollary 3.2.14, there exists a neighborhood $\mathscr{V}_5$ of g_X in $\mathscr{A}_L$ such that each $g \in \mathscr{V}_5$ is hyperbolic far way from the critical point 1. We can choice a neighborhood $\mathscr{V}_2$ of X in $\mathscr{X}^k(M, \partial M)$ in a such away that for all $Y \in \mathscr{V}_2$, the one dimensional map g_Y associate to Y is a element of $\mathscr{V}_5$ because Proposition 3.3.1. Define $\mathscr{V} = \mathscr{V}_1 \cap \mathscr{V}_2$.

Fix $\delta > 0$ and $Y \in \mathscr{V}$. The existence of C_1 and λ_1 (which only depend on Y) and which satisfies the inequality (3.44) is a consequence of the fact that C_H^γ is invariant and contracting by R_Y^{-1}.

Now we will prove the remainder of the Proposition. Indeed, as $g_Y \in \mathscr{V}_5$ we have that there are $C_5 = C_5(\delta, X) > 0$ and $\lambda_5 = \lambda_5(\delta, X) > 1$ such that for all $n \in \mathbb{N}$, for all y with $g_Y^i(y) \in [0, a] \cup [b, 1-\delta]$, $0 \leq i \leq n-1$ we obtain

$$|Dg_Y^n(y)| \geq C_5 \cdot \lambda_5^n. \tag{3.46}$$

Define

$$C_2 = \frac{C_5}{\text{máx}\{\gamma, 1\}} \text{ and } \lambda_2 = \lambda_5.$$

Fix $n \in \mathbb{N}$, $p = (x, y)$ with $R^i(p) \in [0,1] \times ([0, a] \cup [b, 1-\delta])$, $0 \leq i \leq n-1$ and $v \in C_V^\gamma(p)$ with $v = (v_1, v_2)$. Then have that $g_Y^i(y) \in [0, a] \cup [b, 1-\delta]$, $0 \leq i \leq n-1$.

Moreover, from (5.2) we have

$$DR_Y^n(p) = \begin{bmatrix} \partial_x f_Y^n(p) & \partial_y f_Y^n(p) \\ 0 & \partial_y g_Y^n(y) \end{bmatrix}.$$

Therefore, from this equality, the inequality (3.46) and definitions of C_2 and λ_2 we get that

$$\begin{aligned}\| DR_Y^n(p)\cdot v \| &= \text{máx}\{|\partial_x f_Y^n(p)\cdot v_1 + \partial_y f_Y^n(p)\cdot v_2|, |\partial_y g_Y^n(y)|\cdot|v_2|\} \\ &\geq |\partial_y g_Y^n(y)|\cdot|v_2| \\ &\geq \frac{|\partial_y g_Y^n(y)|}{\text{máx}\{\gamma,1\}}\cdot \| v \| \\ &\geq \frac{C_5}{\text{máx}\{\gamma,1\}}\cdot\lambda_5^n\cdot \| v \| \\ &= C_2\cdot\lambda_2^n\cdot \| v \|,\end{aligned}$$

which proof (3.45). The proof follows. ■

Given a curve ζ in Σ we denote by $length(\zeta)$ the length of a curve ζ. We say that ζ is tangent to the γ-cone field C_V^γ if $T_p\zeta$ is contained in $C_V^\gamma(p)$ for all $p \in \zeta$.

For the invariant foliation $\mathscr{F}$ and for $p \in \Sigma$, $\mathscr{F}(p)$ denote the leaf of $\mathscr{F}$ containing p. To the foliation $\mathscr{F}$ we can associate the C^1 map $\Pi^{\mathscr{F}} : \Sigma \to \mathbb{R}$ defined by

$$\Pi^{\mathscr{F}}(p) = \mathscr{F}(p) \cap \{0\} \times \mathbb{R}.$$

Proposition 3.3.4.. *Let $X \in \mathscr{X}^\infty(M,\partial M)$ be exhibiting a generic singular cycle as in Theorem 3.1.1. Then, there exist a neighborhood $\mathscr{W}$ of X in $\mathscr{X}^k(M,\partial M)$ such that: For all $Y \in \mathscr{W}$, for all curve ζ tangent to C_V^γ with $\zeta \cap (\bigcap_{i=0}^{\infty} R_Y^{-i}(\Sigma)) \neq \emptyset$ there exists an integer $n = n(R_Y,\zeta)$ such that $\Pi^{\mathscr{F}}(R_Y^n(\zeta))$ contains $[0,1]$.*

Proof. Fix $X \in \mathscr{X}^\infty(M,\partial M)$ as in the statement of Proposition. Note that X has associate an one dimensional map g_X. From Proposition 3.3.1 (see statements (c)) we get that g_X is a C^3 map, $g_X \in \mathscr{A}_L$ and g_X has negative Schwarzian derivative at [b,1). Then by Corollary 3.2.5, there exists a C^1-neighborhood $\mathscr{U}$ of g_X in $\mathscr{A}_L$ such that for all $g \in \mathscr{U}$ and for all interval J with $J \cap (\bigcap_{i=0}^{\infty} g^{-i}(I)) \neq \emptyset$ there exists $n = n(g,J) \geq 0$ such that $g^n(J) \supset [0,1]$. We can choice a neighborhood $\mathscr{W}$ of X in $\mathscr{X}^k(M,\partial M)$ in a such away that for all $Y \in \mathscr{W}$, the one dimensional map g_Y associate to Y is a element of $\mathscr{U}$ because Proposition 3.3.1.

Fix $Y \in \mathscr{W}$ and a curve ζ tangent to C_V^γ with $\zeta \cap (\bigcap_{i=0}^{\infty} R_Y^{-i}(\Sigma)) \neq \emptyset$.

Define $J = \Pi^{\mathscr{F}}(\zeta)$. Note that $J \cap (\bigcap_{i=0}^{\infty} g_Y^{-i}(I)) \neq \emptyset$. Therefore there exists $n = n(g_Y,J) \geq 0$

such that $g_Y^n(J) \supset [0,1]$. So exists an integer $n = n(R_Y, \zeta)$ such that $\Pi^{\mathscr{F}}(R_Y^n(\zeta)) = g_Y^n\left(\Pi^{\mathscr{F}}(\zeta)\right) = g_Y^n(J) \supset [0,1]$. The Proposition follows. ■

Now we are ready to prove that the set given in (3.35) is transitive for R_Y. Indeed, denote by $R = R_Y$.

Claim A: All stable leaf $L \in \mathscr{F}$ intersecting $\bigcap_{n\geq 0} R^{-n}(H_{0,a} \cup H_{b,1})$ is accumulate by hyperbolic periodic points of saddle type.

Indeed, take a stable leaf $L \in \mathscr{F}$ intersecting $\bigcap_{n\geq 0} R^{-n}(H_{0,a} \cup H_{b,1})$ and take a curve ζ tangent to C_V^{γ} intersecting L. Because $\zeta \cap L \neq \emptyset$ and L is a contracting leaf we have that

$$\zeta \cap \left(\bigcap_{n\geq 0} R^{-n}(H_{0,a} \cup H_{b,1}) \right) \neq \emptyset. \tag{3.47}$$

It follows from (3.47) and Proposition 3.3.4 the existence of a curve $\overline{\zeta} \subseteq \zeta$ and $n(\bar{\zeta}) \in \mathbb{N}$ such that $R^i(\overline{\gamma}) \subseteq H_{0,a} \cup H_{b,1}$ $\forall 0 \leq i \leq n(\overline{\gamma}) - 1$ and $R^{n(\overline{\gamma})}(\bar{\gamma})$ meets all leaf in $\mathscr{F}$.

Let $H(\overline{\zeta})$ be the horizontal band in Σ consisting of saturating $\overline{\zeta}$ by the foliation $\mathscr{F}$. By the property of $\overline{\zeta}$ and $n = n(\overline{\zeta})$ above we have that $R^n(H(\overline{\zeta}))$ crosses $H(\overline{\zeta})$ in a hyperbolic way. Then, by standard index arguments (see [56]) there is a periodic point of R on $R^n(H(\overline{\zeta})) \cap H(\overline{\zeta})$. Such a periodic point is hyperbolic saddle by Proposition 3.3.3. By taking ζ close to L, the band $H(\overline{\zeta})$ remain close to L and then we have that such a point is close to L. This proves the claim A.

Claim B: The hyperbolic periodic points of saddle type of R are dense in $\bigcap_{n\in\mathbb{Z}} R^n(H_{0,a} \cup H_{b,1})$. Indeed, take a point $z \in \bigcap_{n\in\mathbb{Z}} R^n(H_{0,a} \cup H_{b,1})$ and take a neighborhood V of z. We can choice a large enough integer n such that a neighborhood U of the leaf that contains $R^{-n}(z)$, i.e. a small horizontal band around the leaf, be applied into V. By Claim A there exists a periodic point of saddle type in U. Therefore the claim B follows.

To finish the proof of the transitivity of R we will use the classical Birkhoff's criterium to prove transitivity: for all $p, q \in \bigcap_{n\in\mathbb{Z}} R^n(H_{0,a} \cup H_{b,1})$ and $\varepsilon > 0$ there are $z \in \bigcap_{n\in\mathbb{Z}} R^n(H_{0,a} \cup H_{b,1})$ and $n_z \in \mathbb{N}$ such that $d(z,p) < \varepsilon$ and $d(R^{n_z}(z), q) < \varepsilon$. Indeed, fix p, q and ϵ. By the above claim B we can assume that p and q are hyperbolic periodic points of saddle type. Fix a curve γ in $W^u(p)$ contained in $H_{0,a} \cup H_{b,1}$. We can assume that γ intersects to the leaf $\mathscr{F}(q)$ transversally in some point z^* by Proposition 3.3.4. Since the positive (resp.

negative) orbit of z^* is asymptotic to q (resp. p) we have $z^* \in \bigcap_{n\in\mathbb{Z}} R^n(H_{0,a} \cup H_{b,1})$. By taking the negative orbit of z^* we have some $n_1^* \in \mathbb{N}$ such that

$$d(R^{-n_1^*}(z^*), p) < \varepsilon.$$

By taking the positive orbit of z^* we have some $n_2^* \in \mathbb{N}$ such that

$$d(R^{n_2^*}(z^*), q) < \varepsilon.$$

Then $z = R^{-n_1^*}(z^*)$ and $n_z = n_1^* + n_2^*$ works.

Therefore, the proof of Fist step, Second step and Third step follows. So finish the proof of Theorem 3.1.1. ■

Chapter 4

Robust transitive sets of triangular maps

Let Σ be the unit square $[0,1] \times [0,1]$, U be a open set in $\mathbb{R}^2$ which contains Σ and fix two reals number a, b, $0 < a < b < 1$. We will consider the class $\mathcal{T} \subset \{R : [0,1] \times [0,a] \cup [0,1] \times [b,1] \subset \Sigma \to U\}$ of triangular maps having the points $(x,1)$ as criticalities. We give conditions for the existence of a C^1-robust transitive set for maps in $\mathcal{T}$.

4.1. Triangular maps and statement of the Main Theorem.

In the literature, triangular maps are reserved for properly continuous maps in $[0,1] \times [0,1]$ which are skew product, i. e., they preserve the constant vertical foliation (see for instance [9], [30]). A *triangular map* is any $F \in \mathcal{C}(I \times I)$, $I = [0,1]$, such that $F(x,y) = (f(x), g(x,y))$ for any $(x,y) \in I^2$. For a chronological list of authors who contributed to the problem of classification of triangular maps, see Smítal [75]. Moreover, Bautista, Morales [8] also consider certain "triangular maps"defined on a finite disjoint union $\hat{\Sigma}$ of copies of $[-1,1] \times [-1,1]$ and which are discontinuous maps still preserving a continuous (but not necessarily constant) vertical foliation with large domain. Also is assumed hypotheses imposing certain amount of differentiability close to the points whose iterates fall eventually in the interior of $\hat{\Sigma}$. The existence of periodic points for hyperbolic triangular maps that satisfy other additional conditions and have large domain was proved by Bautista, Morales [8].

Let Σ be the unit square $[0,1] \times [0,1]$ and U be a open set in $\mathbb{R}^2$ which contain to Σ and we shall fix two reals number a, b, $0 < a < b < 1$. We will consider the class $\mathcal{T} \subset \{R : [0,1] \times [0,a] \cup [0,1] \times [b,1] \subset \Sigma \to U\}$ of *triangular maps* having the points $(x,1)$ as criticalities. Moreover, this maps preserving a continuous (but not necessarily constant) horizontal foliation. The main result asserting the existence of triangular maps exhibiting $\mathcal{T}$-robust transitive set,i.e. there is an open set $\mathscr{U}$ in $\mathcal{T}$ a such that all $R \in \mathscr{U}$ the maximal R-invariant set en Σ is a transitive set.

The paper is organized in three sections. In first section we introduce the Statement of the main Theorem. In the second section we prove exponential grow of derivatives for maps in an open set. Finally in the last section we prove the main theorem.

4.1.1. Triangular maps

Hereafter $\Sigma = [0,1] \times [0,1]$ denotes the unit close square and $U \subset \mathbb{R}^2$ a open set containing Σ.

Fix two real numbers a, b with $0 < a < b < 1$.

Denote by $p = (x, y) = (x_p, y_p)$ the natural coordinate system in U. Put

$$L_0 = \{y = 0\};\ L_a = \{y = a\};\ L_b = \{y = b\};\ L_1 = \{y = 1\}.$$

We call a closed subset $H \subseteq \Sigma$ (resp. $V \subseteq \Sigma$) a *horizontal* (resp. *vertical*) *band* if it is bounded by two disjoint continuous curves connecting the vertical (resp. horizontal) slides of Σ, $\{(0,y) : 0 \leq y \leq 1\}$ (resp. L_0) and $\{(1,y) : 0 \leq y \leq 1\}$ (resp. L_1). See Figure 4.1.

Given a map F, we denote by Dom(F) the domain of F.

A *curve* c in U is the image of a C^1 injective map $c : Dom(c) \subset \mathbb{R} \to U$ with $Dom(c)$

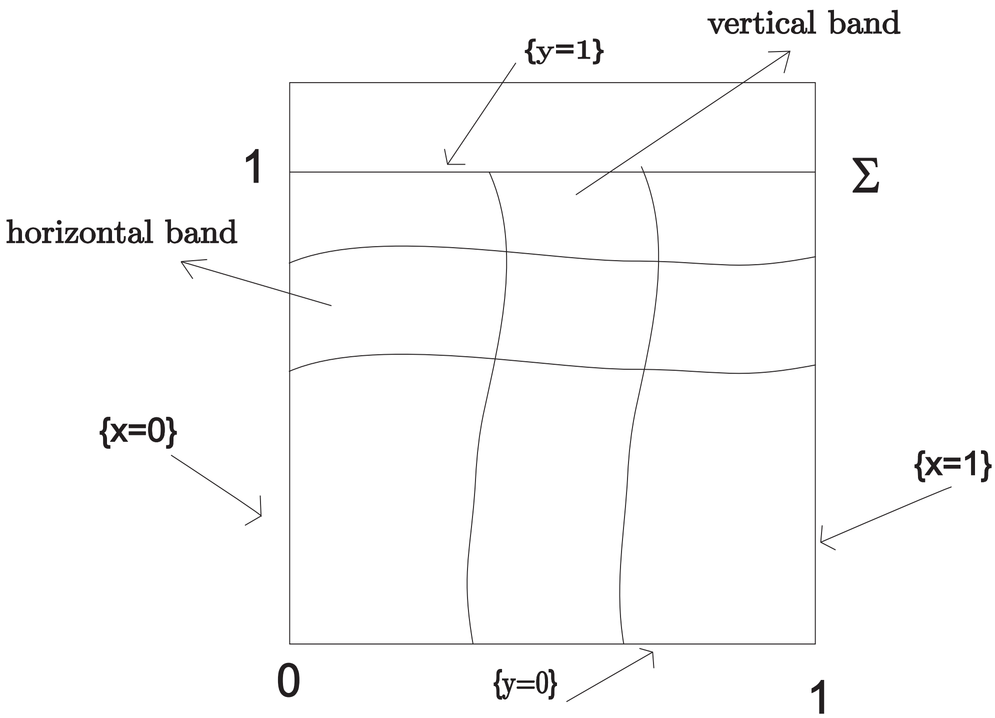

Figura 4.1: Shape of vertical and horizontal band.

being a compact interval. We often identify c with its image set. A curve c is *horizontal* if it is the graph of a C^1 map $h : [0,1] \to U \cap \mathbb{R}$, i.e., $c = \{(x, h(x)) : x \in [0,1]\} \subset U$.

Let M be a differentiable m dimensional manifold, $m > 0$. A foliation $\mathscr{F}$ of dimension n, $0 < n < m$, is a decomposition of M in n dimensional submanifolds, called leaves of the foliation. The foliation $\mathscr{F}$ is C^k, $k \geq 0$, if the holonomy map defined in transversal cross-sections is a C^k map. If $k = 0$ we said that the $\mathscr{F}$ is a *continuous foliation.*

Definition 4.1.1.. *A continuous foliation $\mathscr{F}$ on U is called horizontal if its leaves are horizontal curves and the curves L_0, L_a, L_b, L_1 are leaves of $\mathscr{F}$.*

It follows from the definition above that the leaves of a horizontal foliation $\mathscr{F}$ are horizontal curves hence differentiable ones. In particular, for all leaf L, the tangent space T_pL is well defined for all $p \in L$.

Let H_{y_0,y_1} denote the horizontal band $[0,1] \times [y_0, y_1]$.

Definition 4.1.2.. *Let $R : H_{0,a} \cup H_{b,1} \subset \Sigma \to U$ be a map and $\mathscr{F}$ be a continuous foliation* *on U. We say that R* preserves $\mathscr{F}$ *if for every leaf L of $\mathscr{F}$ contained in $H_{0,a} \cup H_{b,1}$ there is a leaf $\tilde{L}$ of $\mathscr{F}$ such that $R(L) \subset \tilde{L}$). In this case we say that $\mathscr{F}$ is contracting if there are a constante $C > 0$ and $0 < \lambda < 1$ such that*

$$\| DR^n(p) \cdot v \| \leq C \cdot \lambda^n,$$

for all $n \in \mathbb{N}$, $p \in L$, $L \in \mathscr{F}$ and $v \in T_pL$.

Now we can define triangular map.

Definition 4.1.3. (Triangular map). *A map $R : H_{0,a} \cup H_{b,1} \subset \Sigma \to U$ is called triangular if it preserves and contrae a horizontal foliation $\mathscr{F}$. Moreover $R(L_0) \subset ([0,1) \times \{0\})$, $R(L_a) \subset U \backslash \Sigma$, $R(L_b) \subset U \backslash \Sigma$ and $R(L_1) = \{(c_0, 0)\}$ for some $c_0 \in (0,1)$.*

Notation. Let $\mathcal{T}$ denote the set of all the triangular maps R.

4.1.2. Quasi-hyperbolic triangular maps

The Quasi-hyperbolicity will be defined through cone fields in U: Denote by TU the tangent bundle of U. Given $p \in U$ and $\gamma > 0$, we denote by $C^\gamma(p)$ the vertical cone with inclination γ, i.e.,

$$C^\gamma(p) = \{v \in T_pU : v = (u, w); |u| \leq \gamma \cdot |w|\}.$$

A *cone field* in U is a continuous map $C^\gamma : p \in U \mapsto C^\gamma(p) \subset T_pU$, where $C^\gamma(p)$ is a cone with constant inclination γ on T_pU. Let $R : H_{0,a} \cup H_{b,1} \subset \Sigma \to U$ be any differentiable map. A cone field C^γ is called *R-invariant* if $DR(C^\gamma(p) \backslash \{0\}) \subset int(C^\gamma(R(p)))$ for all $p \in H_{0,a} \cup H_{b,1}$. Moreover, C^γ is called *transversal* to a horizontal foliation $\mathscr{F}$ on U if $T_pL \cap C^\gamma(p) = \{0\}$, $\forall p \in L$ and $\forall L \in \mathscr{F}$.

Now we can define quasi-hyperbolic triangular map.

Definition 4.1.4. (Quasi-hyperbolic triangular map). *Let $R : H_{0,a} \cup H_{b,1} \subset \Sigma \to U$ be a triangular map with associated horizontal foliation $\mathscr{F}$. For two maps $\alpha, \tilde{\alpha} : \mathcal{T} \to (1, \infty)$ and numbers K_0, K_1, ν and μ such that $K_0 > 0$, $K_1 > 0$ and $1 < \nu \leq \mu$, we say that R is $(K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha})$-quasi hyperbolic if*

(H1) *R is a* C^1*-diffeomorphisms in* $H_{0,a} \cup (H_{b,1} \setminus \{y = 1\})$.

(H2) ν, μ, $\alpha = \alpha(R)$ *and* $\tilde{\alpha} = \tilde{\alpha}(R)$ *satisfy:* $\nu \cdot \mu^{\frac{1-\tilde{\alpha}}{\alpha}} > 1$.

(H3) $y_{R(p)} \leq K_0 \cdot \mid y_p - 1 \mid^{\alpha}, \qquad \forall p \in H_{b,1}$.

(H4) *There are* $0 < \gamma < \frac{1}{2}$ *and an invariant cone field* C^{γ} *in* U *transverse to* $\mathscr{F}$ *such that:*

(H4-a) $\forall p \in H_{b,1}, \forall v \in C^{\gamma}(p)$;

$$\| DR(p) \cdot v \| \geq K_1 \cdot \mid y_p - 1 \mid^{\tilde{\alpha}-1} \cdot \| v \| .$$

(H4-b) $\forall p \in H_{0,a}, \forall v \in C^{\gamma}(p)$

$$\nu \cdot \| v \| \leq \| DR(p) \cdot v \| \leq \mu \cdot \| v \| .$$

Notation. Let $\tilde{\mathcal{T}}$ denote the set of all the maps R which are $(K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha})$-quasi-hyperbolics.

Figure 4.2 displays the essential features of the map $R \in \tilde{\mathcal{T}}$.

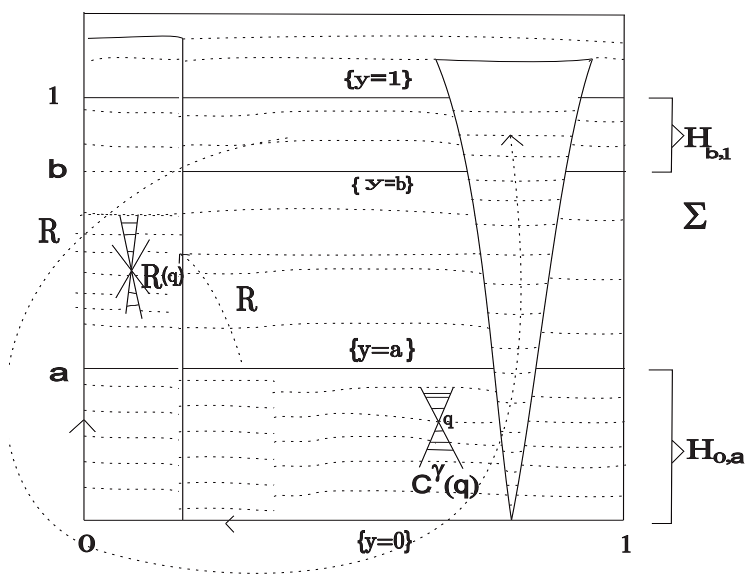

Figura 4.2: Shape of R.

4.1.3. Definition of the C^1-topology in $\tilde{\mathcal{T}}$

Definition 4.1.5.. *In the space of maps $\tilde{\mathcal{T}}$ we consider the C^1-topology which is defined by the metric*

$$\begin{aligned} d_{C^1}(R,\hat{R}) &= \text{máx}\Big\{ \sup_p \| R(p)-\hat{R}(p) \|, \sup_p \| DR(p)-D\hat{R}(p) \|, \\ & \quad | \alpha(R)-\alpha(\hat{R}) |, | \tilde{\alpha}(R)-\tilde{\alpha}(\hat{R}) | : p \in H_{0,a} \cup H_{b,1} \Big\}. \end{aligned}$$

4.1.4. Schwarzian derivative

Definition 4.1.6.. *Let $f : Dom(f) \subset \mathbb{R} \to \mathbb{R}$ be a C^3 map. The Schwarzian derivative of f at $x \in dom(f)$ with $Df(x) \neq 0$ is defined as*

$$Sf(x) = \frac{D^3 f(x)}{Df(x)} - \frac{3}{2} \cdot \Big(\frac{D^2 f(x)}{Df(x)} \Big)^2.$$

We say that f has negative Schwarzian derivative si $Sf(x) < 0$ for all $x \in Dom(f)$ such that $Df(x) \neq 0$.

From the definition 4.1.6, the following formula for the Schwarzian derivative of the composition of two C^3 maps follows immediately by the chain rule,

$$S(g \circ f)(x) = Sg(f(x)) \cdot | Df(x) |^2 + Sf(x).$$

Hence, the Schwarzian derivative of the iterates of f is given by

$$Sf^n(x) = \sum_{i=0}^{n-1} Sf(f^i(x)) \cdot | Df^i(x) |^2 .$$

Therefore, if a map has negative Schwarzian derivative, so do all its iterates.

The lemma below (see [42], pg. 154) is the main analytical property of maps of negative Schwarzian derivative that will be used in the section 4.2.

Lemma 4.1.7. (Minimum Principle). *Let T be a closed interval with end points r, s and $f : T \subset dom(f) \to \mathbb{R}$ be a map with negative Schwarzian derivative. If $Df(x) \neq 0$ for all $x \in T$, then*

$$| Df(x) | > \text{mín}\{| Df(r) |, | Df(s) |\}, \qquad \forall x \in (r, s).$$

4.1.5. Hypothesis (H)

We impose some regularity on a horizontal foliation $\mathscr{F}$ associate to a triangular map.

To any horizontal foliation $\mathscr{F}$ we can associate the *holonomy map* $\Pi^{\mathscr{F}} : U \to \mathbb{R}$ defined by

$$\Pi^{\mathscr{F}}(p) = \mathscr{F}(p) \cap \{0\} \times \mathbb{R},$$

where $\mathscr{F}(p)$ is the leaf of $\mathscr{F}$ through the point p.

A R-invariant horizontal foliation $\mathscr{F}$ can be used to define a new coordinate system

$$(\bar{x}, \bar{y}) = \varphi(x, y) = (x, \Pi^{\mathscr{F}}(x, y)) \tag{4.1}$$

in a such way that $\bar{R} = \varphi \circ R \circ \varphi^{-1}$ has the following shape:

$$\bar{R}(\bar{x}, \bar{y}) = (g(\bar{x}, \bar{y}), f(\bar{y})) \tag{4.2}$$

holds for some maps $f : Dom(f) \subset \mathbb{R} \to \mathbb{R}$ and $g : dom(g) \subset \mathbb{R}^2 \to \mathbb{R}$.

Definition 4.1.8. (linear-contracting map). *Let $f : [0, a] \cup [b, 1] \subset \mathbb{R} \to \mathbb{R}$ be a C^3 map. We say that its is a linear-contracting if:*

(h1) There exists $\rho_f > 1$ such that $f(x) = \rho_f \cdot x$ for all $x \in [0, a]$. Moreover, f is decreasing on $[b, 1]$, $f(1) = 0$ and $Df(x) = 0$ if and only if $x = 1$. Additionally $f(a) > 1$ and $f(b) > 1$.

(h2) f has negative Schwarzian derivative on $[b, 1)$.

Figure 4.3 displays the essential features of a linear-contracting map f.

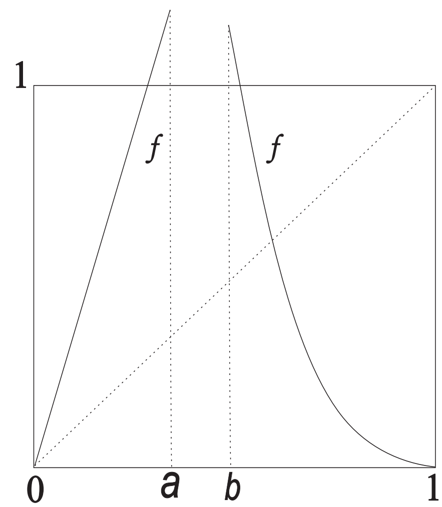

Figura 4.3: Shape of f.

Definition 4.1.9. (Hypothesis (H)). *Let $R : H_{0,a} \cup H_{b,1} \subset \Sigma \to U$ be a triangular map with associate horizontal foliation $\mathscr{F}$ of class C^3. We say that R satisfies (H) if the map f given by (4.2) is linear-contracting and $\Pi^{\mathscr{F}}$ additionally satisfies the property:*

$$\left|\frac{\partial \Pi^{\mathscr{F}}}{\partial x}(x,y)\right| < \frac{1}{2} \text{ and } \left|\frac{\partial \Pi^{\mathscr{F}}}{\partial y}(x,y)\right| > \frac{3}{4}.$$

Our main theorem is the following result.

Theorem 4.1.10. (Main Theorem). *Let R_0 be a $(K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha})$-quasi-hyperbolic map (i,e., $R_0 \in \tilde{\mathcal{T}}$) satisfying (H). Then there exists a C^1-neighborhood $\mathscr{U} = \mathscr{U}(R_0)$ of R_0 in $\tilde{\mathcal{T}}$ such that for all $R \in \mathscr{U}$, the maximal invariant set,*

$$\bigcap_{n \in \mathbb{Z}} R^n(\Sigma) \tag{4.3}$$

be transitive, *i.e., there is z on it such that $\{R^n(z) : n \in \mathbb{N}\}$ is dense in the maximal invariant set given by (4.3).*

4.2. Elementary results

We start with the following lemma.

Lemma 4.2.1.. *Let R_0 be a triangular map with associate horizontal foliation $\mathscr{F}$ of class C^1 such that $\Pi^{\mathscr{F}}$ satisfies $\left|\frac{\partial \Pi^{\mathscr{F}}}{\partial x}(x,y)\right| < \frac{1}{2}$ and $\left|\frac{\partial \Pi^{\mathscr{F}}}{\partial y}(x,y)\right| > \frac{3}{4}$. Let $(\bar{x},\bar{y}) = \varphi(x,y)$ be the coordinates given by (4.1) and f be the map as in (4.2). Then, there exists a constant $\hat{C}_0 = \hat{C}_0(R_0) > 0$ such that for all $i \in \mathbb{N}$, for all $p \in dom(R_0^i)$ and for all $v \in C^\gamma(p)$,*

$$\| DR_0^i(p) \cdot v \| \geq \hat{C}_0 \cdot \mid Df^i(\bar{y}) \mid \cdot \| v \|, \tag{4.4}$$

where $(\bar{x},\bar{y}) = \varphi(p)$.

Proof. Fix R_0 as in the statement of the lema. Let consider the coordinates $(\bar{x},\bar{y})$ given by (4.1):

$$(\bar{x},\bar{y}) = \varphi(x,y) = (x, \Pi^{\mathscr{F}}(x,y)). \tag{4.5}$$

So, $\bar{R}_0 = \varphi \circ R_0 \circ \varphi^{-1}$ became the map

$$\bar{R}_0 = (g(\bar{x},\bar{y}), f(\bar{y})) \tag{4.6}$$

according (4.2).

We define

$$M = \text{máx}\{\| D\varphi(\hat{z}) \| : \hat{z} \in dom(\varphi)\}.$$

and

$$\hat{C}_0 = \frac{1}{2 \cdot M}.$$

To continue we claim

Claim A. For all $\bar{p} = (\bar{x},\bar{y}) \in dom(\bar{R}_0)$ and for all $\bar{v} = (\bar{u},\bar{w}) \in \mathbb{R}^2$ we obtain

$$\| D\bar{R}_0(\bar{p}) \cdot \bar{v} \| \geq \mid Df(\bar{y}) \mid \cdot \mid \bar{w} \mid .$$

Indeed, fix $\bar{p} = (\bar{x},\bar{y}) \in dom(\bar{R}_0)$ and $\bar{v} = (\bar{u},\bar{w}) \in \mathbb{R}^2$.

From (4.6) and the definition of Jacobian matrix we obtain

$$D\bar{R}_0(\bar{p}) \cdot \bar{v} = \begin{pmatrix} \frac{\partial g}{\partial \bar{x}}(\bar{p}) \cdot \bar{u} + \frac{\partial g}{\partial \bar{y}}(\bar{p}) \cdot \bar{w} \\ Df(\bar{y}) \cdot \bar{w} \end{pmatrix}.$$

In this equality, we denote by $\tilde{u} = \frac{\partial g}{\partial \bar{x}}(\bar{p}) \cdot \bar{u} + \frac{\partial g}{\partial \bar{y}}(\bar{p}) \cdot \bar{w}$ and $\tilde{w} = Df(\bar{y}) \cdot \bar{w}$. Therefore, considering the maximum norm, we obtain

$$\| D\bar{R}_0(\bar{p}) \cdot \bar{v} \| \; = \; \text{máx}\{\mid \tilde{u} \mid, \mid \tilde{w} \mid\}. \tag{4.7}$$

If $\mid \tilde{u} \mid \geq \mid \tilde{w} \mid$ then of (4.7) and the of the definition of $\tilde{w}$ we have

$$\begin{aligned} \| D\bar{R}_0(\bar{p}) \cdot \bar{v} \| &= \text{máx}\{\mid \tilde{u} \mid, \mid \tilde{w} \mid\} \\ &= \mid \tilde{u} \mid \\ &\geq \mid \tilde{w} \mid \\ &= \mid Df(\bar{y}) \mid \cdot \mid \bar{w} \mid . \end{aligned}$$

On the other hand, if $\mid \tilde{u} \mid < \mid \tilde{w} \mid$ then of (4.7) and the of the definition of $\tilde{w}$ we get that

$$\begin{aligned} \| D\bar{R}_0(\bar{p}) \cdot \bar{v} \| &= \text{máx}\{\mid \tilde{u} \mid, \mid \tilde{w} \mid\} \\ &= \mid \tilde{w} \mid \\ &= \mid Df(\bar{y}) \mid \cdot \mid \bar{w} \mid . \end{aligned}$$

This proof the Claim A.

Claim B. For all $j \in \mathbb{N}$, for all $\bar{p} = (\bar{x}, \bar{y}) \in dom(\bar{R}_0^j)$ and for all $\bar{v} = (\bar{u}, \bar{w}) \in \mathbb{R}^2$ we have that

$$\| D\bar{R}_0^j(\bar{p}) \cdot \bar{v} \| \geq \mid Df^j(\bar{y}) \mid \cdot \mid \bar{w} \mid . \tag{4.8}$$

Indeed, using induction on j and the Claim A, the proof of (4.8) of Claim B follows.

Now fix $i \in \mathbb{N}$, $p \in dom(R_0^i)$ and $v \in C^\gamma(p)$. We denote by $p = (x, y)$ and $v = (u, w)$.

From (4.5) and definition of Jacobian matrix we obtain

$$D\varphi(p) \cdot v = \begin{pmatrix} u \\ \frac{\partial \Pi^{\mathscr{F}}}{\partial x}(p) \cdot u + \frac{\partial \Pi^{\mathscr{F}}}{\partial y}(p) \cdot w \end{pmatrix}.$$

We denote $\bar{u} = u$, $\bar{w} = \frac{\partial \Pi^{\mathscr{F}}}{\partial x}(x, y) \cdot u + \frac{\partial \Pi^{\mathscr{F}}}{\partial y}(x, y) \cdot w$ and $\varphi(p) = \bar{p} = (\bar{x}, \bar{y})$.

For other hand, as $\varphi \circ \varphi^{-1} = Id$ then for all $\bar{q} \in dom(\varphi^{-1})$ and for all $\bar{z} \in \mathbb{R}^2$ we have that

$$\begin{aligned} \| \bar{z} \| &= \| D\varphi(\varphi^{-1}(\bar{q})) \cdot D\varphi^{-1}(q) \cdot \bar{z} \| \\ &\leq \| D\varphi(\varphi^{-1}(\bar{q})) \| \cdot \| D\varphi^{-1}(\bar{q}) \cdot \bar{z} \| \end{aligned}$$

therefore we obtain

$$\| D\varphi^{-1}(\bar{q}) \cdot \bar{z} \| \geq \frac{1}{M} \cdot \| \bar{z} \| . \tag{4.9}$$

Remember that $R_0 = \varphi^{-1} \circ \bar{R}_0 \circ \varphi$. Then from Chain Rule Theorem and definition of $\bar{v}$ we obtain

$$\begin{aligned} DR_0^i(p) \cdot v &= D\varphi^{-1}(\bar{R}_0^i(\varphi(p))) \cdot D\bar{R}_0^i(\varphi(p)) \cdot D\varphi(p) \cdot v \\ &= D\varphi^{-1}(\bar{R}_0^i(\varphi(p))) \cdot D\bar{R}_0^i(\varphi(p)) \cdot \bar{v}. \end{aligned} \tag{4.10}$$

From (4.10), the definition of $\bar{p}$, the inequality (4.9) (taking $\bar{q} = \bar{R}_0^i(\varphi(p))$ and $\bar{z} = D\bar{R}_0^i(\bar{p}) \cdot \bar{v}$) and (4.8) of Claim B we get that

$$\begin{aligned} \| DR_0^i(p) \cdot v \| &= \| D\varphi^{-1}(\bar{R}_0^i(\varphi(p))) \cdot D\bar{R}_0^i(\bar{p}) \cdot \bar{v} \| \\ &\geq \frac{1}{M} \cdot \| D\bar{R}_0^i(\bar{p}) \cdot \bar{v} \| \\ &\geq \frac{1}{M} \cdot | Df^i(\bar{y}) | \cdot | \bar{w} | . \end{aligned} \tag{4.11}$$

Then using that $\bar{w} = \frac{\partial \Pi^{\mathscr{F}}}{\partial x}(p) \cdot u + \frac{\partial \Pi^{\mathscr{F}}}{\partial y}(p) \cdot w$, the bounds for the partial derivatives of $\Pi^{\mathscr{F}}$ and the fact that $0 < \gamma < \frac{1}{2}$, we obtain that

$$\begin{aligned} | \bar{w} | &= \Big| \frac{\partial \Pi^{\mathscr{F}}}{\partial x}(p) \cdot u + \frac{\partial \Pi^{\mathscr{F}}}{\partial y}(p) \cdot w \Big| \\ &= | w | \cdot \Big| \frac{\partial \Pi^{\mathscr{F}}}{\partial x}(p) \cdot \frac{u}{w} + \frac{\partial \Pi^{\mathscr{F}}}{\partial y}(p) \Big| \\ &\geq | w | \cdot \Big(- \Big| \frac{\partial \Pi^{\mathscr{F}}}{\partial x}(p) \Big| . \Big| \frac{u}{w} \Big| + \Big| \frac{\partial \Pi^{\mathscr{F}}}{\partial y}(p) \Big| \Big) \\ &\geq | w | \cdot \Big(- \frac{1}{2} \cdot \gamma + \frac{3}{4} \Big) \\ &\geq | w | \cdot \Big(- \frac{1}{2} . \frac{1}{2} + \frac{3}{4} \Big) \\ &= \frac{1}{2} \cdot | w | . \end{aligned} \tag{4.12}$$

Therefore, (4.11) and (4.12) imply that

$$\| DR_0^i(p) \cdot v \| \geq \frac{1}{2 \cdot M} \cdot \mid Df^i(\bar{y}) \mid \cdot \mid w \mid$$

By hypothesis, $0 < \gamma < \frac{1}{2} < 1$ and $v \in C^{\gamma}(p)$. So

$$\begin{aligned} \| v \| &= \text{máx}\{\mid u \mid, \mid w \mid\} \\ &\leq \text{máx}\{\gamma \cdot \mid w \mid, \mid w \mid\} \\ &= \mid w \mid \cdot \text{máx}\{\gamma, 1\} \\ &= \mid w \mid, \end{aligned}$$

then this inequality, the preceding estimates and definition of $\hat{C}_0$ imply that

$$\begin{aligned} \| DR_0^i(p) \cdot v \| &\geq \frac{1}{2 \cdot M} \cdot \mid Df^i(\bar{y}) \mid \cdot \| v \| \\ &= \hat{C}_0 \cdot \mid Df^i(\bar{y}) \mid \cdot \| v \|, \end{aligned}$$

and this proves (4.4) of Lemma 4.2.1. ■

The Minimum Principle given by Lemma 4.1.7 will be used to find an lower bound, non depending on i, for the derivative $Df^i(x)$ for all x such that $f^i(x)$ is far from 1 and 0.

Lemma 4.2.2..*Let $f : [0, a] \cup [b, 1] \to \mathbb{R}$ be a linear-contracting map and let $\tilde{c}$ and $\tilde{d}$ be two real numbers with $0 < \tilde{c} < \tilde{d} < 1$. Then there exists a constant $\tilde{C}_0 = \tilde{C}_0(f, \tilde{c}, \tilde{d}) > 0$ such that for all $i \in \mathbb{N}$ and for all $x \in (0, a) \cup (b, 1)$, if $f^i(x) \in [\tilde{c}, \tilde{d}]$, then*

$$|Df^i(x)| \geq \tilde{C}_0. \tag{4.13}$$

Proof. Fix $f : [0, a] \cup [b, 1] \to \mathbb{R}$ and $\tilde{c}$ and $\tilde{d}$ as in lemma.

Define

$$\tilde{C}_0 = \text{mín}\{\tilde{c}, 1 - \tilde{d}\}.$$

Let consider a interval $J \subset \mathbb{R}$. We denote by *lenght*(J) the length of J.

Now, fix $i \in \mathbb{N}$ and $x \in (0, a) \cup (b, 1)$ such that $f^i(x) \in [\tilde{c}, \tilde{d}]$.

Let us consider $I_x = [\xi_0, \xi_1]$ the maximal interval containing x where f^i is defined. For maximality of I_x we have that either $[0, \tilde{c}] \subset f^i([\xi_0, x])$ and $[\tilde{d}, 1] \subset f^i([x, \xi_1])$, or $[\tilde{d}, 1] \subset f^i([\xi_0, x])$ and $[0, \tilde{c}] \subset f^i([x, \xi_1])$.

In both cases, by Mean Valued Theorem there are $\tilde{\xi_0}, \tilde{\xi_1}$ with $\xi_0 < \tilde{\xi_0} < x < \tilde{\xi_1} < \xi_1$ such that

$$\mid Df^i(\tilde{\xi_0}) \mid \; = \; \frac{lenght(f^i([\xi_0, x]))}{lenght([\xi_0, x])} \geq lenght(f^i([\xi_0, x])) \geq \tilde{C}_0 \tag{4.14}$$

and

$$\mid Df^i(\tilde{\xi_1}) \mid \; = \; \frac{lenght(f^i([x, \xi_1]))}{lenght([x, \xi_1])} \geq lenght(f^i([x, \xi_1])) \geq \tilde{C}_0, \tag{4.15}$$

Using the Minimum Principle (Lemma 4.1.7), the definition of f restricted to $[0, a]$, the fact $S(f^i)(y) = S(f^{i-1}|(b,1) \circ f|(0,a))(y) < 0$ for all $y \in (0, a)$, inequalities (4.14) and (4.15) we obtain either

$$|Df^i(x)| > \text{mín}\left\{|Df^i(\tilde{\xi_0})|, |Df^i(\tilde{\xi_1})|\right\} \geq \tilde{C}_0,$$

or

$$|Df^i(x)| = \beta_f^i > 1 > \tilde{C}_0.$$

Figure 4.4 it illustrates the situation for f^i with $i = 2$.

Therefore, the proof follows. ■

Lemma 4.2.3.. *Let R_0 be a $(K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha})$-quasi-hyperbolic map (i,e., $R_0 \in \tilde{\mathcal{T}}$) satisfying (H) and let c and d be two real numbers with $0 < c < d < 1$. Then there exists a constant $C_0 = C_0(R_0, c, d) > 0$ such that for every $i \in \mathbb{N}$ and every $p \in dom(R_0^i)$, if $R_0^i(p) \in [0,1] \times [c, d]$, then*

$$\| DR_0^i(p) \cdot v \| \geq C_0 \cdot \| v \| \tag{4.16}$$

for all $v \in C^\gamma(p)$.

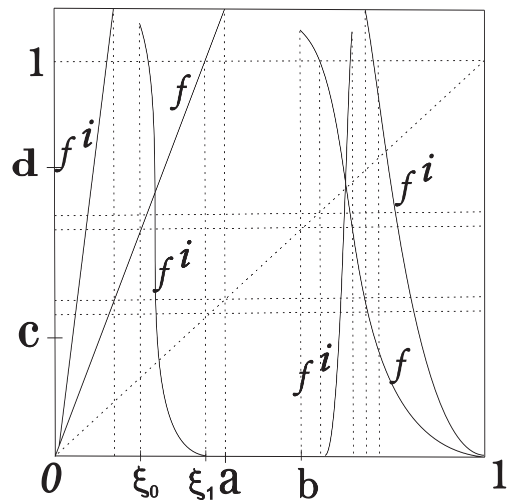

Figura 4.4: Case f^i for $i = 2$.

Proof. Fix R_0, c and d as in the statement of the lemma. Also consider the R_0-invariant foliation $\mathscr{F}$ and the coordinates $(\bar{x}, \bar{y})$ defined by $\mathscr{F}$ in (4.1) and $\bar{R}_0$ given by (4.2).

From Hypothesis (H), the quotient map f according (4.2) has negative Schwarzian derivative.

Take two real numbers $\tilde{c}, \tilde{d}$, with $0 < \tilde{c} < \tilde{d} < 1$ such that

$$z = (x, y) \in [0, 1] \times [c, d] \Longrightarrow \bar{y} \in [\tilde{c}, \tilde{d}]. \tag{4.17}$$

Take $\tilde{C}_0$ given by Lemma 4.2.2 applied to f, $\tilde{c}$ and $\tilde{d}$ as above, and take $\hat{C}_0$ given by Lemma 4.2.1.

Taking

$$C_0 = \hat{C}_0 \cdot \tilde{C}_0$$

we prove that C_0 works. For this we fix $i \in \mathbb{N}$, $p = (x, y) \in dom(R_0^i)$ with $R_0^i(p) \in [0, 1] \times [c, d]$ and $v \in C^\gamma(p)$.

Note that for (4.17), $f^i(\bar{y}) \in [\tilde{c}, \tilde{d}]$, so Lemma 4.2.2 applied to $\bar{y}$ give us

$$\mid Df^i(\bar{y}) \mid \geq \tilde{C}_0. \tag{4.18}$$

Therefore, from (4.4) of Lemma 4.2.1, (4.18) and definition of C_0 we obtain

$$\parallel DR_0^i(p) \cdot v \parallel \geq C_0 \cdot \parallel v \parallel,$$

and the proof follows. ■

Lemma 4.2.4.. *Let R_0 be a $(K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha})$-quasi-hyperbolic map (i,e., $R_0 \in \tilde{\mathcal{T}}$) satisfying (H) and let c_1 and d_1 two real numbers with $0 < c_1 < d_1 < 1$. Then there is $C_1 = C_1(R_0, c_1, d_1) > 0$ such that for each $N \in \mathbb{N}$ there exists a C^1-neighborhood $\mathscr{V}_1 = \mathscr{V}_1(R_0, N, c_1, d_1)$ of R_0 in $\tilde{\mathcal{T}}$ such that for all $l \leq N$, for all $R \in \mathscr{V}_1$ and for all $p \in dom(R^l)$, if $R^l(p) \in [0,1] \times [c_1, d_1]$, then*

$$\| DR^l(p) \cdot v \| \geq C_1 \cdot \| v \|, \tag{4.19}$$

for all $v \in C^\gamma(p)$.

Proof. Fix R_0 and c_1 and d_1 as in the statement of the lemma. Take the real numbers c and d such that $0 < c < c_1$ and $d_1 < d < 1$. It follow from the definition of the C^1-topology of $\tilde{\mathcal{T}}$ that for all $j \geq 1$ there is a C^1-neighborhood $\bar{\mathscr{V}}(j)$ of R_0 in $\tilde{\mathcal{T}}$ such that if $R \in \bar{\mathscr{V}}(j)$, $q \in dom(R^j)$ and $R^j(q) \in [0,1] \times [c_1, d_1]$ then $q \in dom(R_0^j)$ and

$$R_0^j(q) \in [0,1] \times [c, d]. \tag{4.20}$$

To see this, we extend the maps of $\tilde{\mathcal{T}}$ to maps as it is shown in the figure 4.5.

Take C_0 given by Lemma 4.2.3 applied to R_0, c and d as above. It follows again from the definition of the C^1-topology of $\tilde{\mathcal{T}}$ that for all $i \geq 1$ there is a C^1-neighborhood $\tilde{\mathscr{V}}(i)$ of R_0 in $\tilde{\mathcal{T}}$ such that if $R \in \tilde{\mathscr{V}}(i)$, $\tilde{q} \in \Sigma$ and $\tilde{v} \in C^\gamma(\tilde{q})$ then

$$\| DR_0^i(\tilde{q}) \cdot \tilde{v} - DR^i(\tilde{q}) \cdot \tilde{v} \| \leq \frac{C_0}{2} \cdot \| \tilde{v} \| . \tag{4.21}$$

Define

$$C_1 = \frac{C_0}{2}.$$

Now fix an integer $N \geq 1$. Define

$$\bar{\mathscr{V}} = \bar{\mathscr{V}}(f, N) = \bigcap_{1 \leq j \leq N} \bar{\mathscr{V}}(j)$$

and

$$\tilde{\mathscr{V}} = \tilde{\mathscr{V}}(f, N) = \bigcap_{1 \leq i \leq N} \tilde{\mathscr{V}}(i).$$

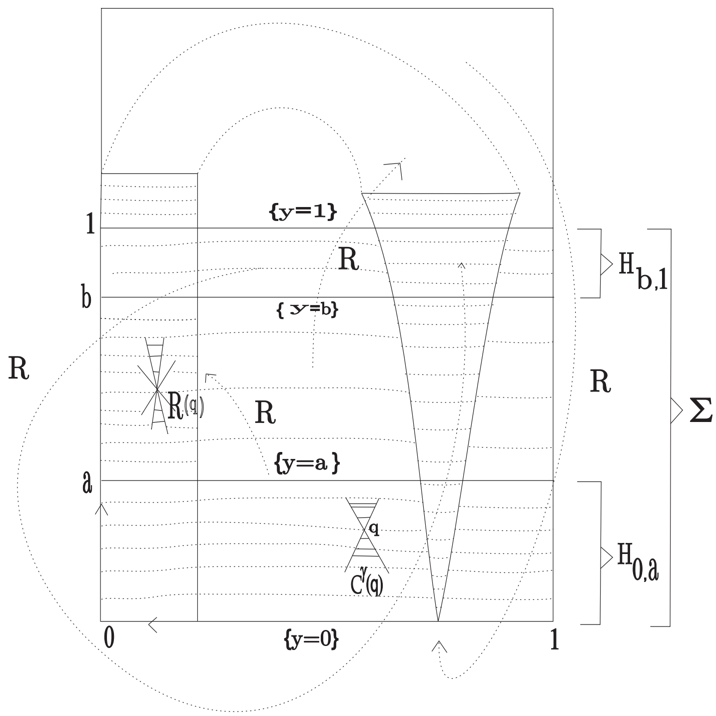

Figura 4.5: Shape of extension of R.

Define

$$\mathscr{V}_1 = \bar{\mathscr{V}} \cap \tilde{\mathscr{V}}.$$

Let us prove that the neighborhood $\mathscr{V}_1$ works. For this we fix an integer $1 \leq l \leq N$, $R \in \mathscr{V}_1$ and $p \in dom(R^l)$ such that $R^l(p) \in [0,1] \times [c_1, d_1]$ and $v \in C^\gamma(p)$. In particular $R \in \bar{\mathscr{V}}$, then (4.20) implies that $R_0^l(p) \in [0,1] \times [c,d]$ (taking $j = l$), then (4.16) in Lemma 4.2.3 (taking $i = l$) implies

$$\| DR_0^l(p) \cdot v \| \geq C_0 \cdot \| v \| . \tag{4.22}$$

Moreover, as in particular $R \in \tilde{\mathscr{V}}$, then (4.21) implies (taking $i = l$, $\tilde{q} = p$ and $\tilde{v} = v$)

$$\| DR_0^l(p) \cdot v - DR^l(p) \cdot v \| \leq \frac{C_0}{2} \cdot \| v \| . \tag{4.23}$$

From (4.23), (4.22) and definition of C_1 we obtain

$$\begin{aligned}
\| DR^l(p)\cdot v \| &= \| DR^l(p)\cdot v - DR_0^l(p)\cdot v + DR_0^l(p)\cdot v \| \\
&\geq \| DR_0^l(p)\cdot v \| - \| DR^l(p)\cdot v - DR_0^l(p)\cdot v \| \\
&\geq C_0\cdot \| v \| - \frac{C_0}{2}\cdot \| v \| \\
&= \frac{C_0}{2}\cdot \| v \| \\
&= C_1\cdot \| v \|
\end{aligned}$$

and the lemma follows. ■

Let $R \in \mathcal{T}$ and $\delta > 0$. Now, we define the auxiliar sets

$$\begin{aligned}
V(\delta) &= [0,1]\times([0,a])\cup[b,1-\delta]); \\
W_R^k(\delta) &= \left\{p=(x,y)\in V(\delta) : R^i(p)\in V(\delta), i=0,...,k-1\right\}, \ \forall k\geq 1; \\
\Lambda_R(\delta) &= \bigcap_{k\geq 1} W_R^k(\delta).
\end{aligned}$$

This set is called the R-maximal invariant set on $V(\delta)$.

Definition 4.2.5.. *Let $R \in \mathcal{T}$ and $\delta > 0$. We said that R is* vertically expansive *on $V(\delta)$ if there are a vertical invariant cone field C^γ on $V(\delta)$, a positive constant $C = C(R,\delta) > 0$ and $\lambda = \lambda(R,\delta) > 1$ such that if $n \in \mathbb{N}$ and $p \in dom(R^{n-1})$ satisfy $R^i(p) \in V(\delta)$ for every $i = 0, \cdots, n-1$ then*

$$\| DR^n(p)\cdot v \| > C\cdot\lambda^n\cdot \| v \|$$

for all $v \in C^\gamma(p)$.

Lemma 4.2.6.. *Let R_0 be a triangular map which satisfies the hypothesis (H). Then for each $\delta > 0$, R_0 is vertical expansive on $V(\delta) = [0,1]\times([0,a])\cup[b,1-\delta])$.*

Proof. Fix R_0 as in the statement of the lemma. Let consider the R_0-invariant foliation $\mathscr{F}$. By definition of the $(\bar{x},\bar{y})$ coordinates, $\bar{R}_0 = \varphi\circ R_0\circ\varphi^{-1}$ according (4.2) became

$$\bar{R}_0(\bar{x},\bar{y}) = (g(\bar{x},\bar{y}), f(\bar{y}))$$

where f has negative Schwarzian derivative by Hypothesis (H).

Also fix $\delta > 0$. Let us consider $\bar{\delta} = \bar{\delta}(\delta) > 0$ in such way that

$$z = (x, y) \notin ([0,1] \times (1-\delta, 1]) \Longrightarrow \bar{y} \notin (1-\bar{\delta}, 1]. \tag{4.24}$$

From (4.4) of Lemma 4.2.1, there exists $\hat{C} > 0$ that for all $k \in \mathbb{N}$, for all $p \in dom(R_0^k)$ and for all $v \in C^\gamma(p)$ we get

$$\| DR_0^k(p) \cdot v \| \;\; \geq \;\; \hat{C} \cdot \mid Df^k(\bar{y}) \mid \cdot \| v \| \tag{4.25}$$

where $\varphi(p) = (\bar{x}, \bar{y})$.

By Singer's and Misiurewicz's Theorems (see [42]) we have that f is hyperbolic on $[0,a] \cup [b, 1-\bar{\delta}]$ there are positive constants $\bar{C} = \bar{C}(\bar{R}_0, \bar{\delta}) > 0$ and $\bar{\lambda} = \bar{\lambda}(\bar{R}_0, \bar{\delta}) > 1$ such that for all $k \in \mathbb{N}$ and $f^i(\bar{y}) \in [0,a] \cup [b, 1-\bar{\delta}]$, $0 \leq i \leq k-1$ we get that

$$\mid Df^k(\bar{y}) \mid \geq \bar{C} \cdot \bar{\lambda}^k. \tag{4.26}$$

Take $C = \hat{C} \cdot \bar{C}$ and $\lambda = \bar{\lambda}$. Fix $k \in \mathbb{N}$ and p with $R_0^i(p) \in V(s), 0 \leq i \leq k-1$. So, from (4.24) we have that $f^i(\bar{y}) \in [0,a] \cup [b, 1-\bar{\delta}]$, $1 \leq i \leq k-1$ because $R_0^i(p) \in V(s), 0 \leq i \leq k-1$. Note that $(\bar{x}, \bar{y}) = \varphi(p)$ because (4.24).

Take $v \in C^\gamma(p)$, then using (4.25), (4.26) and definition of C and λ we obtain

$$\| DR_0^k(p) \cdot v \| \geq C \cdot \lambda^k \cdot \| v \| .$$

Therefore, from Definition 4.2.5, the proof follows. ■

In the following proposition, we show an easy characterization of vertical expansivity in the terms of the maximal invariant set, like for one dimensional hyperbolic set.

Proposition 4.2.7.. *Let $R \in \mathcal{T}$ and $\delta > 0$. Then, R is vertically expansive on $V(\delta)$ if only if there is a vertical invariant cone field C^γ on $V(\delta)$ such that for every $p \in \Lambda_R(\delta)$ and $\forall v \in C^\gamma(p)$ there exists a positive integer $k = k(p, v)$ such that*

$$\| DR^k(p) \cdot v \| > \| v \| .$$

Proof. If R is vertically expansive on $([0,1] \times [0,a]) \cup ([0,1] \times [b, 1-\delta])$ we take k so that $C \cdot \lambda^k > 1$. So let us prove the reverse implication. So suppose that $\| DR^{k(p,v)}(p)v \| > \| v \|$

for all $p \in \Lambda_R(\delta)$ and $\forall v \in C^\gamma(p)$. Denote by B the set of $v \in C^\gamma(p)$ such that $\| v \| = 1$. By compactness of $\Lambda_R(\delta) \times B$ and continuity of the derivative of R, there exists a finite cover $V_1 \times B_1, \cdots, V_k \times B_k$ of $\Lambda_R(\delta) \times B$ by open sets, integers $n_1, \cdots, n_k$ and number $\lambda_1, \cdots, \lambda_k > 1$, such that $\| DR^{n_i}(p) \cdot v \| > \lambda_i$ for all $(p, v) \in V_i \times B_i$ and every $i = 1, \cdots, k$.

Let consider a neighborhood $V = \bigcup_{j=1}^{k} V_j$. Note that exists n_0 such that if $p \notin V$ then there exists i smaller n_0 with $R^i(p) \notin V(\delta)$. Define $\tilde{n} = \text{máx}\{n_j : 0 \leq j \leq k\}$, $a = \text{mín}\{\| DR(p) \cdot v \| : (p, v) \in V(\delta) \times B\}$, $\lambda = \text{mín}\{\sqrt[n_i]{\lambda_i} : 0 < i \leq k\}$ and $C = \text{mín}\{\frac{a^i}{\lambda^i} : 1 \leq i \leq \tilde{n}\}$.

Take $n \in \mathbb{N}$ and $p \in dom(R^{n-1})$ such that $R^i(p) \notin [0,1] \times (1-\delta, 1]$ for every $i = 0, \cdots, n-1$. Take $v \in C^\gamma(p)$.

We follows inductively the following alternatives:

a) If $p \notin V$. Then $n \leq n_0 \leq \tilde{n}$. So

$$\| DR^n(p) \cdot v \| = \| DR^n(p) \cdot \frac{v}{\| v \|} \| \cdot \| v \| > C \cdot \lambda^n \| v \| .$$

b) If $p \in V$. Then, there is i, $1 \leq i \leq k$ such that $(p, \frac{v}{\|v\|}) \in V_i \times B_i$.

b-1) If $n < n_i$ then $n \leq \tilde{n}$. So

$$\| DR^n(p) \cdot v \| = \| DR^n(p) \cdot \frac{v}{\| v \|} \| \cdot \| v \| > C \cdot \lambda^n \| v \| .$$

b-2) If $n \geq n_i$ then

$$\begin{aligned} \| DR^n(p) \cdot v \| &= \| DR^{n-n_i}(R^{n_i}(p)) \cdot \frac{DR^{n_i}(p) \cdot v}{\| DR^{n_i}(p) \cdot v \|} \| \cdot \\ &\quad \| DR^{n_i}(p) \cdot \frac{v}{\| v \|} \| \cdot \| v \| \\ &\geq \| DR^{n-n_i}(q) \cdot w \| \cdot \lambda^{n_i} \cdot \| v \| \end{aligned}$$

where $q = R^{n_i}(p)$ and $w = \frac{DR^{n_i}(p) \cdot v}{\|DR^{n_i}(p) \cdot v\|}$. Then the proof follows recursively. ■

Remark 4.2.8.. *It is clear from definition that Vertical expansiveness is an C^1-open property. In our case, Vertical expansiveness is equivalent to hiperbolicity because the existence of an invariant contracting foliation.*

Lemma 4.2.9. *Let R_0 be a $(K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha})$-quasi-hyperbolic map (i,e., $R_0 \in \tilde{\mathcal{T}}$) satisfying (H). Then there are $\delta_2 = \delta_2(R_0) > 0$ and a constant $C_2 = C_2(R_0) > 0$ satisfying the following property: for each $\delta < \delta_2$, there are $\lambda_2 = \lambda_2(R_0, \delta) > 1$ and a C^1-neighborhood $\mathscr{V}_2 = \mathscr{V}_2(R_0, \delta)$ of R_0 in $\tilde{\mathcal{T}}$ such that for all $R \in \mathscr{V}_2$, for all $k \in \mathbb{N}$ and for all $p \in dom(R^k)$ with $p, R(p), ..., R^{k-1}(p) \in V(\delta)$ but $R^k(p) \in [0,1] \times [1-\delta_2, 1]$, then*

$$\| DR^k(p) \cdot v \| \geq C_2 \cdot \lambda_2^k \cdot \| v \|, \tag{4.27}$$

for all $v \in C^\gamma(p)$.

Proof. Let consider R_0 as in statement of the Lemma. Choose $\delta_2 > 0$, c_1, d_1 with $0 < c_1 < d_1 < 1$ and a C^1-neighborhood $\bar{\mathscr{V}_2}$ of R_0 in $\tilde{\mathcal{T}}$ in such way that if $R \in \bar{\mathscr{V}_2}$ and $p \in dom(R)$ satisfy $R(p) \in [0,1] \times [1-\delta_2, 1]$ then $p \in [0,1] \times [c_1, d_1]$.

Let C_1 be as in Lemma 4.2.4 applied to R_0, c_1 and d_1 chosen above. Let consider $\tilde{C}_1 < \text{ínf}\{\frac{\|DR_0(p)\cdot v\|}{\|v\|} : p \in [0,1] \times [c_1, d_1], v \in C^\gamma(p)\}$. Shrinking $\bar{\mathscr{V}_2}$, we can suppose that for all $R \in \bar{\mathscr{V}_2}$, for all $p \in [0,1] \times [c_1, d_1]$ and for all $v \in C^\gamma(p)$, $\| DR(p) \cdot v \| > \tilde{C}_1 \cdot \| v \|$.

Define

$$C_2 = \text{mín}\{1, \frac{C_1.\tilde{C}_1}{2}\}.$$

Now fix δ, $0 < \delta < \delta_2$. For such a δ we shall take $\mathscr{V}_2$ and λ_2 as follows:

From Lemma 4.2.6, R_0 is vertical expansive on $V(\delta) = [0,1] \times ([0,a] \cup [b, 1-\delta])$, so we can find a C^1-neighborhood $\tilde{\mathscr{V}} = \tilde{\mathscr{V}}(R_0, \delta)$ of R_0 in $\tilde{\mathcal{T}}$ and constants $\tilde{C} = \tilde{C}(R_0, \delta)$, $\tilde{\lambda} = \tilde{\lambda}(R_0, \delta)$, with $\tilde{C} > 0$ and $\lambda > \tilde{\lambda} > 1$ such that if $R \in \tilde{\mathscr{V}}$ and p satisfy $R^i(p) \in V(\delta)$, $0 \leq i \leq k-1$ and $v \in C^\gamma(p)$, then

$$\| DR^k(p) \cdot v \| \geq \tilde{C} \cdot \tilde{\lambda}^k \cdot \| v \|, \tag{4.28}$$

because this is an open property (see Remark 4.2.8).

From (4.28) we can find $K = K(R_0, \delta) \in \mathbb{N}$ and $\hat{\lambda}_2 = \hat{\lambda}_2(R_0, \delta)$, $\tilde{\lambda} > \hat{\lambda}_2 > 1$ such that if $k \geq K$, $g \in \tilde{\mathscr{V}}$ and p satisfy $R^i(p) \in V(\delta)$, $0 \leq i \leq k-1$ and $v \in C^\gamma(p)$, then

$$\| DR^k(p) \cdot v \| \geq \hat{\lambda}_2^k \cdot \| v \| \, . \tag{4.29}$$

(Just take $K = \text{mín}\{k : \tilde{C} \cdot \tilde{\lambda}^k > 1\}$ and $\hat{\lambda}_2$ such that $1 < \hat{\lambda}_2 < \text{mín}\{\tilde{\lambda} \cdot \tilde{C}^{\frac{1}{K}}, \tilde{\lambda}\}$).

Let $\mathscr{V}_1$ be the C^1-neighborhood of R_0 in $\tilde{\mathcal{T}}$ given by Lemma 4.2.4 for this K.

Let us consider $\lambda_2 = \lambda_2(R_0, \delta)$, $1 < \lambda_2 < \hat{\lambda}_2$ such that

$$\lambda_2^K < 2. \tag{4.30}$$

We show that Lemma work with $\mathscr{V}_2 = \mathscr{V}_2(R_o, \delta) = \bar{\mathscr{V}_2} \cap \tilde{\mathscr{V}} \cap \mathscr{V}_1$ and λ_2 as was chosen.

Fix $R \in \mathscr{V}_2$, $k \in \mathbb{N}$ and $p \in dom(R^k)$ with $p, R(p), ..., R^{k-1}(p) \in V(\delta)$ but $R^k(p) \in [0,1] \times [1-\delta_2, 1]$. Also fix $v \in C^\gamma(p)$.

If $k \geq K$, then (4.29), the definition of λ_2 and C_2 imply that

$$\| DR^k(p) \cdot v \| \geq \hat{\lambda}_2^k \cdot \| v \| \geq C_2 \cdot \lambda_2^k \cdot \| v \| . \tag{4.31}$$

To the case when $k < K$, first we observe that $R^{k-1}(p)$ belong to $[0,1] \times [c_1, d_1]$ because $R^k(p) \in [0,1] \times [1-\delta_2, 1]$ and by hypothesis $R \in \mathscr{V}_2$. Then, definition of $\tilde{C}_1$ for one side and (4.19) of Lemma 4.2.4 imply that

$$\begin{aligned} \| DR^k(p) \cdot v \| &= \| DR(R^{k-1}(p)) \cdot DR^{k-1}(p) \cdot v \| \\ &\geq \tilde{C}_1 \cdot \| DR^{k-1}(p) \cdot v \| \\ &\geq \tilde{C}_1 \cdot C_1 \cdot \| v \| . \end{aligned}$$

So,

$$\| DR^k(p) \cdot v \| > C_2 \cdot \lambda_2^k \tag{4.32}$$

because (4.30) and the definitions of C_2 and λ_2.

Finally, from (4.31) and (4.32) the lemma follows. ∎

Lemma 4.2.10.. *Let R_0 be a $(K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha})$-quasi-hyperbolic map (i,e., $R_0 \in \tilde{\mathcal{T}}$) and let C such that $0 < C \leq 1$. Then there are a C^1-neighborhood $\mathscr{V}_3 = \mathscr{V}_3(R_0, C)$ of R_0 in $\tilde{\mathcal{T}}$ and constants $\delta_3 = \delta_3(R_0, C) > 0$, $\lambda_3 = \lambda_3(R_0, C) > 1$ and $L = L(R_0, C) \in \mathbb{N}$ with $C \cdot \lambda_3^L > 1$ such that for each $R \in \mathscr{V}_3$, for each $p \in [0,1] \times [1-\delta_3, 1)$ there exists an integer $l = l(R, p) > L$ such that $R^j(p) \in [0,1] \times [0, a]$ for $j = 1, ..., l-1$ and*

$$\| DR^l(p) \cdot v \| \geq \lambda_3^l \cdot \| v \| \tag{4.33}$$

for all $v \in C^\gamma(p)$.

Proof. Fix R_0, $K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha}$ and $0 < C < 1$ as in statement of the lemma. For every $\eta > 0$ we consider the C^1-neighborhood for R_0 of size η in $\tilde{\mathcal{T}}$, that is

$$\mathscr{V}_\eta = \{R : d_{C^1}(R_0, R) < \eta\}.$$

By (H2) we have $\nu \cdot \mu^{\frac{1-\tilde{\alpha}(R_0)}{\alpha(R_0)}} > 1$. Then there is $\hat{\lambda} = \hat{\lambda}(R_0) > 1$ and $\eta > 0$ small such that for all $R \in \mathscr{V}_\eta$

$$\nu \cdot \mu^{\frac{1-\tilde{\alpha}(R)}{\alpha(R)}} > \hat{\lambda} > 1. \tag{4.34}$$

As $R_0(\{y = 1\}) \subseteq \{y = 0\}$ we can choose $0 < \hat{\delta} = \hat{\delta}(R_0) < 1$ such that $R_0(p) \in [0, 1] \times [0, \frac{a}{2}]$ for all $p \in [0, 1] \times (1 - \hat{\delta}, 1)$. Shrinking η again, we can assume that $R(p) \in [0, 1] \times [0, a]$ for all $p \in [0, 1] \times (1 - \hat{\delta}, 1)$ and for all $R \in \mathscr{V}_\eta$.

For $R \in \mathscr{V}_\eta$ and $p \in [0, 1] \times [1 - \hat{\delta}, 1)$ we define

$$l(R, p) = \text{ínf}\{j \geq 1 : R^j(R(p)) \notin H_{0,a}\}. \tag{4.35}$$

To choose λ_3 we need to make some estimatives. Let consider $R \in \mathscr{V}_\eta$ and $p \in [0, 1] \times [1 - \hat{\delta}, 1)$ and $v \in C^\gamma(p)$. By definition of l in (4.35) we have that $R(p), ..., R^{l-1}(p) \in [0, 1] \times [0, a]$ and

$$y_{R^{l+1}(p)} > a. \tag{4.36}$$

Claim. For all $R \in \tilde{\mathcal{T}}$, for all $q \in \Sigma = [0, 1] \times [0, 1]$ and $n \in \mathbb{N}$ with q, $R(q)$, $R^2(q), ..., R^{n-1}(q) \in H_{0,a}$ then $y_{R^n(q)} \leq \mu^n \cdot y_q$.

Indeed, fix $R \in \tilde{\mathcal{T}}$, $q \in \Sigma$ and $n \in \mathbb{N}$ such that q, $R(q)$, $R^2(q), ..., R^{n-1}(q) \in H_{0,a}$.

By definition we have

$$y_{R^n(q)} = (\Pi_y \circ R^n)(q)$$

were Π_y is the projection over the second variable y. Define for $t \in [0, 1]$ the real valued map $h(t) = y_{R^n(x_q,t)}$, then

$$h(y_q) = y_{R^n(x_q,y_q)}.$$

By Mean Valued Theorem we have

$$y_{R^n(x_q,y_q)} = h'(\xi).y_q \tag{4.37}$$

for some ξ because $y_{R^n(x_q,0)} = 0$.

But

$$h'(\xi) = D(\Pi_y \circ R^n)(x_q, \xi) \cdot (0,1).$$

By (H4-b) because $(0,1) \in C^\gamma(x,y)$, $\forall (x,y) \in H_{0,a}$ with $R(x,y)$, ..., $R^{n-1}(x,y) \in H_{0,a}$, the cone C^γ is invariant and the fact that $\| \Pi_y \| = 1$ we get

$$\begin{aligned} \mid h'(\xi) \mid &\leq \| \Pi_y \| \cdot \| DR^n(x_q,\xi).(0,1) \| \\ &\leq \mu^n \end{aligned}$$

and then (4.37) applies and proves the claim.

Moreover, for all $R \in \mathscr{V}_\eta$ and all $p \in [0,1] \times [1-\hat{\delta}, 1)$, the above Claim (for $q = R(p)$ and $n = l$) implies that

$$\begin{aligned} y_{R^{l+1}(p)} &= y_{R^l(R(p))} \\ &\leq \mu^l \cdot y_{R(p)}. \end{aligned}$$

From this inequality and (4.36) we get

$$y_{R(p)} > a \cdot \mu^{-l}. \tag{4.38}$$

But (H3) says

$$y_{R(p)} \leq K_0 \cdot \mid y_p - 1 \mid^{\alpha(R)} . \tag{4.39}$$

Note that by definition of the neighborhood $\mathscr{V}_\eta$,

$$\alpha(R_0) - \eta < \alpha_R < \alpha(R_0) + \eta$$

and

$$\tilde{\alpha}(R_0) - \eta < \tilde{\alpha}(R) < \tilde{\alpha}(R_0) + \eta.$$

By definition of the neighborhood $\mathscr{V}_\eta$, as $p \in [0,1] \times [1-\hat{\delta}, 1)$, (4.38) and (4.39) we obtain

$$K_0 \cdot (\hat{\delta})^{\alpha(R_0)-\eta} \geq a \cdot \mu^{-l}.$$

Therefore,

$$l \geq \frac{log(a) - log(K_0) - (\alpha(R_0)-\eta) \cdot log(\hat{\delta})}{log(\mu)} = L(\hat{\delta}). \tag{4.40}$$

Note that $L(\hat{\delta}) \to \infty$ as $\hat{\delta} \to 0$.

Also, from (4.38) and (4.39) we have

$$\begin{aligned} \mid y_p - 1 \mid^{\tilde{\alpha}(R)-1} &\geq \left(\frac{y_{R(p)}}{K_0}\right)^{\frac{\tilde{\alpha}(R)-1}{\alpha(R)}} \\ &\geq \left(\frac{a}{K_0}\right)^{\frac{\tilde{\alpha}(R)-1}{\alpha(R)}} \cdot \left(\mu^{\frac{1-\tilde{\alpha}(R)}{\alpha(R)}}\right)^l. \end{aligned} \tag{4.41}$$

From the Chain Rule, (H4-a) and (H4-b) we get

$$\begin{aligned} \parallel DR^l(p) \cdot v \parallel &= \parallel DR^{l-1}(R(p)) \cdot DR(p)v \parallel \\ &\geq \nu^{l-1} \cdot \parallel DR(p) \cdot v \parallel \\ &\geq \nu^{l-1} \cdot K_1 \cdot \mid y_p - 1 \mid^{\tilde{\alpha}(R)-1} \cdot \parallel v \parallel, \end{aligned} \tag{4.42}$$

for all $v \in C^\gamma(p)$.

Moreover, by C^1 proximity we have

$$\frac{\tilde{\alpha}(R_0) - 1 - \eta}{\alpha(R_0) + \eta} < \frac{\tilde{\alpha}(R) - 1}{\alpha(R)} < \frac{\tilde{\alpha}(R_0) - 1 + \eta}{\alpha(R_0) - \eta}. \tag{4.43}$$

Using successively (4.42), (4.41), (4.34) and (4.43) we obtain that for all $v \in C^\gamma(p)$,

$$\begin{aligned} \parallel DR^l(p) \cdot v \parallel &\geq \nu^{l-1} \cdot K_1 \cdot \left(\frac{a}{K_0}\right)^{\frac{\tilde{\alpha}(R)-1}{\alpha(R)}} \cdot \left(\mu^{\frac{1-\tilde{\alpha}_R}{\alpha_R}}\right)^l \cdot \parallel v \parallel \\ &= \frac{K_1}{\nu} \cdot \left(\frac{a}{K_0}\right)^{\frac{\tilde{\alpha}(R)-1}{\alpha(R)}} \cdot \left(\nu\mu^{\frac{1-\tilde{\alpha}(R)}{\alpha(R)}}\right)^l \cdot \parallel v \parallel \\ &\geq \frac{K_1}{\nu} \cdot \left(\frac{a}{K_0}\right)^{\frac{\tilde{\alpha}(R)-1}{\alpha(R)}} \cdot \hat{\lambda}^l \cdot \parallel v \parallel \\ &= C(R_0) \cdot \hat{\lambda}^l \cdot \parallel v \parallel \end{aligned} \tag{4.44}$$

where $C(R_0) = \text{mín}\left\{\frac{K_1}{\nu}\cdot\left(\frac{a}{K_0}\right)^{\frac{\tilde{\alpha}(R_0)-1-\eta}{\alpha(R_0)+\eta}}, \frac{K_1}{\nu}\cdot\left(\frac{a}{K_0}\right)^{\frac{\tilde{\alpha}(R_0)-1+\eta}{\alpha(R_0)-\eta}}\right\}$.

Now, fix $L_0 \in \mathbb{N}$ such that $C(R_0)\cdot\hat{\lambda}^{L_0} > 1$. Also take λ_3 such that

$$1 < \lambda_3 < min\{(C(R_0))^{\frac{1}{L_0}}\cdot\hat{\lambda}, \hat{\lambda}\}.$$

By (4.40) we take $\hat{\delta}$ such that for all $p \in [0,1]\times[1-\hat{\delta},1)$ and $R \in \mathscr{V}_\eta$ we have $L(\hat{\delta}) > L_0$. Therefore, $l = l(R,p) \geq L_0$.

So, using inequality (4.44) and definition of λ_3 we obtain

$$\begin{aligned} \| DR^l(p)\cdot v \| &\geq C(R_0)\cdot\hat{\lambda}^{L_0}\cdot\hat{\lambda}^{l-L_0}\cdot \| v \| \\ &\geq \lambda_3^{L_0}\cdot\lambda_3^{l-L_0}\cdot \| v \| \\ &= \lambda_3{}^{l}\cdot \| v \|, \end{aligned}$$

for all $v \in C^\gamma(p)$.

Finally take $L = L(f,C)$ with $C\cdot\lambda_3^L > 1$. Shrinking $\hat{\delta}$ in such a way $L(\hat{\delta}) > L$. This implies that for all $p \in [0,1]\times[1-\delta,1)$ and for all $R \in \mathscr{V}_\eta$, $l = l(R,p) > L$.

The lemma works with $\mathscr{V}_3 = \mathscr{V}_\eta$, λ_3, L and $\delta_3 = \hat{\delta}$ as before. This ends the proof. ■

Proposition 4.2.11.. *Let R_0 be a $(K_0,K_1,\nu,\mu,\alpha,\tilde{\alpha})$-quasi-hyperbolic map (i,e., $R_0 \in \tilde{\mathcal{T}}$) satisfying (H). Then there are C^1-neighborhood $\mathscr{V}_4 = \mathscr{V}_4(R_0)$ of R_0 in $\tilde{\mathcal{T}}$ and constants $C_4 = C_4(R_0) > 0$, $\delta_4 = \delta_4(R_0) > 0$ and $\lambda_4 = \lambda_4(R_0) > 1$ satisfying the following properties: If $k \in \mathbb{N}$, $R \in \mathscr{V}_4$, $p \in dom(R^k)$ are such that $R^k(p) \in [0,1]\times(1-\delta_4,1]$ and $v \in C^\gamma(p)$ then*

$$\| DR^k(p)\cdot v \| \geq C_4\cdot\lambda_4^k\cdot \| v \| . \tag{4.45}$$

Moreover, if $p \in [0,1]\times(1-\delta_4,1)$ then

$$\| DR^k(p)\cdot v \| \geq \lambda_4^k\cdot \| v \| . \tag{4.46}$$

Proof. Fix R_0 as in lemma. Let us consider $C_2 > 0$ and δ_2 given in Lemma 4.2.9 applied for R_0.

Take $C_4 = \text{mín}\{1, C_2\}$. Applying Lemma 4.2.10 for R_0 and $C = C_4$ we obtain a C^1-neighborhood $\mathscr{V}_3$ and the real numbers δ_3 and λ_3 and an integer L. Choose δ_4 such that

$0 < \delta_4 = \frac{1}{2} \cdot \text{mín}\{\delta_2, \delta_3\}$. Take λ_2 and $\mathscr{V}_2$ given by Lemma 4.2.9 applied to $\delta = \delta_4$. Let us consider $\mathscr{V}_4 = \mathscr{V}_2 \cap \mathscr{V}_3$ and choose λ_4 in a such way that $1 < \lambda_4 < \text{mín}\{C_2^{\frac{1}{L}} \cdot \lambda_3, \lambda_2\}$. Note that $C_2^{\frac{1}{L}} \cdot \lambda_3 > 1$ because $C_2 > C = C_4$.

Now we prove that the proposition works with $\mathscr{V}_4$, C_4, δ_4 and λ_4 chosen above.

Fix $R \in \mathscr{V}_4$, $k \in \mathbb{N}$, $p \in dom(R^k)$ such that $R^k(p) \in [0,1] \times (1-\delta_4, 1]$ and $v \in C^\gamma(p)$.

We decompose the orbit $\{R^i(p)\}_{i=0}^k$ in several blocks as follows:

$$\{p = p_1, R(p_1), ..., R^{k_1-1}(p_1)\}, \{q_1 = R^{k_1}(p_1), R(q_1), ..., R^{l_1-1}(q_1)\},$$

$$\{p_2 = R^{l_1}(q_1), R(p_2), ..., R^{k_2-1}(p_2)\}, \{q_2 = R^{k_2}(p_2), R(y_2), ..., R^{l_2-1}(q_2)\}, \cdots,$$

$$\{p_m = R^{l_{m-1}}(q_{m-1}), R(p_m), ..., R^{k_m}(p_m) = q_m = R^k(p)\},$$

where k_1 is the first integer such that $R^{k_1}(p_1) \in (1-\delta_4, 1)$, $l_1 \geq L$ is given by the conclusion of Lemma 4.2.10 applied to q_1, k_2 is the first integer that $R^{k_2}(p_2) \in (1-\delta_4, 1)$ and so on.

Notice that $k_1 + l_1 + \cdots + k_{m-1} + l_{m-1} + k_m = k$.

Using the Chain Rule Theorem, (4.27) of Lemma 4.2.9, (4.33) of Lemma 4.2.10, and the definitions of C_4 and λ_4 we obtain

$$\begin{aligned}
\| DR^k(p)v \| &= \| DR^{k_m}(p_m) \cdot DR^{l_{m-1}}(q_{m-1}) \ldots DR^{k_3}(p_3) \cdot DR^{l_2}(q_2) \cdot \\
&\quad DR^{k_2}(p_2) \cdot DR^{l_1}(q_1) \cdot Dg^{k_1}(p_1)v \| \\
&\geq \left((C_2 \cdot \lambda_2^{k_m}) \cdot \lambda_3^{l_{m-1}}\right) \ldots \left((C_2 \cdot \lambda_2^{k_3}) \cdot \lambda_3^{l_2}\right) \cdot \left((C_2 \cdot \lambda_2^{k_2}) \cdot \lambda_3^{l_1}\right) \cdot \\
&\quad (C_2 \cdot \lambda_2^{k_1}) \cdot \| v \| \\
&= \left(\lambda_2^{k_m}(C_2 \cdot \lambda_3^{l_{m-1}})\right) \ldots \left(\lambda_2^{k_3} \cdot (C_2 \cdot \lambda_3^{l_2})\right) \cdot \left(\lambda_2^{k_2} \cdot (C_2 \cdot \lambda_3^{l_1})\right) \cdot \\
&\quad (C_2 \cdot \lambda_2^{k_1}) \cdot \| v \| \\
&\geq (\lambda_2^{k_m} \cdot \lambda_4^{l_{m-1}}) \cdots (\lambda_2^{k_3} \cdot \lambda_4^{l_2}) \cdot (\lambda_2^{k_2} \cdot \lambda_4^{l_1}) \cdot (C_2 \cdot \lambda_2^{k_1}) \cdot \| v \| \\
&= \lambda_2^{k_2 + \cdots + k_m} \cdots \lambda_4^{l_1 + \cdots + l_{m-1}} \cdot (C_2 \cdot \lambda_2^{k_1}) \cdot \| v \| \qquad (4.47) \\
&\geq C_2 \cdot \lambda_4^k \cdot \| v \| \\
&\geq C_4 \cdot \lambda_4^k \cdot \| v \|,
\end{aligned}$$

this proves (4.45) of Proposition 4.2.11.

For finish the proof note that if $p \in [0,1] \times (1-\delta_4, 1)$ then in the decomposition of the orbit $\{R^i(p)\}_{i=0}^k$ given above, k_1 does not exist, so in (4.47), the expression $C_2 \cdot \lambda_2^{k_1}$ does not appears. Therefore, (4.46) of Proposition 4.2.11 follows. This concludes the proof . ■

Corollary 4.2.12.. *Let R_0 be a $(K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha})$-quasi-hyperbolic map (i,e., $R_0 \in \tilde{\mathcal{T}}$) satisfying (H). Then there exists a C^1-neighborhood $\mathscr{V}_5$ of R_0 en $\tilde{\mathcal{T}}$ such that for each $R \in \mathscr{V}_5$ and for each $\delta > 0$, R is vertically expansive on $[0,1] \times ([0,a] \cup [b, 1-\delta])$.*

Proof. Fix R_0 as in the statement of the lemma. Let us consider δ_4 and the C^1-neighborhood $\mathscr{V}_4$ given by Proposition 4.2.11. Because $R_0 \in \tilde{\mathcal{T}}$ which satisfies (H) implies that R_0 is vertically expansive on $[0,1] \times ([0,a] \cup [b, 1-\delta_4])$ (see Lemma 4.2.6). From Remark 4.2.8, we can find a C^1-neighborhood $\tilde{\mathscr{V}_4} = \tilde{\mathscr{V}_4}(R_0)$ of R_0 in $\tilde{\mathcal{T}}$ such that all $R \in \tilde{\mathscr{V}_4}$ is vertically expansive on $[0,1] \times ([0,a] \cup [b, 1-\delta_4])$. Define $\mathscr{V}_5 = \mathscr{V}_4 \cap \tilde{\mathscr{V}_4}$. Now take $R \in \mathscr{V}_5$ and δ with $0 < \delta < \delta_4$. We will prove that R is vertical expansive on $[0,1] \times [b, 1-\delta]$. In order to do this, let consider p in the maximal R-invariant set in $[0,1] \times ([0,a] \cup [b, 1-\delta])$ and $v \in C^\gamma(p)$. Then, we have that either $\forall k > 1, R^k(p) \notin [0,1] \times (1-\delta_4, 1]$ or for some $k_1 > 1, R^{k_1}(p) \in [0,1] \times (1-\delta_4, 1]$. In the first case, taking k big enough we have that $\| DR^k(p) \cdot v \| > \| v \|$ because $R \in \tilde{\mathscr{V}_4}$. In the other case, by (4.45) of Proposition 4.2.11, $\| DR^{k_1}(p) \cdot v \| \geq C_4 \cdot \lambda_4^{k_1} \cdot \| v \|$.

Now applying the same argument to $R^{k_1}(p)$, we have two alternatives: there is k big enough such that $\| DR^{k_1+k}(p) \cdot v \| > \| v \|$ or there is k_2 such that $R^{k_2}(R^{k_1}(p)) \in [0,1] \times (1-\delta_4, 1)$ in a such case by (4.46) of Proposition 4.2.11 we have that

$$\| DR^{k_2+k_1}(p) \cdot v \| \geq C_4 \cdot \lambda_4^{k_1} \cdot \lambda_4^{k_2} \cdot \| v \| \,.$$

Inductively, we obtain that for some $k = k(R, \delta, n, p, v) \in \mathbb{N}$ big enough such that $\| DR^k(p) v \| > \| v \|$.

Therefore, from Proposition 4.2.7 the proof follows. ■

4.3. Proof of the Main Theorem

In this section we prove the Main Theorem (Theorem 4.1.10).

Let $R : H_{0,a} \cup H_{b,1} \subset \Sigma \to U$. A point $p \in H_{0,a} \cup H_{b,1}$ is *periodic* for R if there is $n \geq 1$ such that $R^j(p) \in H_{0,a} \cup H_{b,1}$ for all $0 \leq j \leq n-1$ and $R^n(p) = p$. The *ω-limit set* of a point $p \in \bigcap_{i=0}^{\infty} R^{-i}(\Sigma)$ is the set

$$\{q \in U : q = \lim_{k\to\infty} R^{n_k}(p) \, \text{for some sequence} \, n_k \to \infty\}.$$

The *basin* of a periodic point p is the set of points whose $\omega-$limit set contains p. We say that a periodic point p of period n is *sink* if its basin contain an open set.

Given a curve ζ in Σ we denote by *length*(ζ) the length of a curve ζ. We say that γ is *tangent* to the cone field C^γ if $T_p\zeta$ is contained in $C^\gamma(p)$ for all $p \in \zeta$.

If $\mathscr{F}$ is a continuous foliation on U and $A \subset U$ then the *saturated* of A for $\mathscr{F}$ it is union of leaves of $\mathscr{F}$ which pass through points of A and will denoted by $[A]$.

Theorem 4.3.1.. *Let R_0 be a $(K_0, K_1, \nu, \mu, \alpha, \tilde{\alpha})$-quasi-hyperbolic map (i,e., $R_0 \in \tilde{\mathscr{T}}$) satisfying (H). Then there exists a C^1-neighborhood $\mathscr{V} = \mathscr{V}(R_0)$ of R_0 in $\tilde{\mathscr{T}}$ such that for all $R \in \mathscr{V}$, the maximal R-invariant set contained in $\Sigma = [0,1] \times [0,1]$, $\Lambda_R = \bigcap_{i=0}^{\infty} R^{-i}(\Sigma)$, don't contain a curve tangent to C^γ.*

Proof. Fix R_0 as in statement of the theorem. Let us consider $\mathscr{V}_4$, δ_4 and λ_4 given by Proposition 4.2.7. Let us consider the C^1-neighborhood $\mathscr{V}_5$ of R_0 in $\tilde{\mathscr{T}}$ given by Corollary 4.2.12. Take $\mathscr{V} = \mathscr{V}_4 \cap \mathscr{V}_5$. Now fix $R \in \mathscr{V}$.

Suppose, by contradiction, $\Lambda_R = \bigcap_{i=0}^{\infty} R^{-i}(\Sigma)$, has a curve ζ tangent to C^γ.

We split the proof in some steps. Let consider the invariant foliation $\mathscr{F}$ associated to the triangular map R. Also, we denote by g the quotient map induced by $\mathscr{F}$.

Step 1. R has no sinks. Indeed, Corollary 4.2.12 and the fact that the foliation $\mathscr{F}$ is contracting imply that all the periodic points p are saddle like hyperbolic point(see Remark 4.2.8). Therefore, R has no sinks.

Step 2. For all $m \neq n$, $[R^m(\zeta)] \cap [R^n(\zeta)]$ has no interior. Indeed, if there are integers $m \neq n$ such that $[R^m(\zeta)] \cap [R^n(\zeta)]$ has non-empty interior. Now we denote by $J = \Pi^{\mathscr{F}}(\zeta)$ and it is clear that J is a homterval for g (i.e., $g^n|J$ is a homeomorphism for all $n \in \mathbb{N}$). Therefore

$g^m(J) \cap g^n(J)$ has non-empty interior, then by standard arguments (see [23], Lemma A, pag. 142) we obtain that g has a sink. Furthermore as the foliation $\mathscr{F}$ is contracting we obtain that R has a sink. This is a contradiction with step 1.

Therefore, the sequence of horizontal bands $\{[R^n(\zeta)]\}_{n=0}^{\infty}$ are pairwise disjoint and can not accumulate a sink, i.e. ζ is a "wandering curve". From this it follows that

Step 3.

$$lengh(R^n(\zeta)) \to 0 \text{ as } n \to +\infty. \tag{4.48}$$

Indeed, suppose that (4.48) is not valid , then there exists a real number $\beta > 0$ such that

$$lenght(R^n(\zeta)) > \beta.$$

for infinitely n.

Let $\Sigma^2 = \Sigma \times \Sigma$. Consider the compact $K \subset \Sigma^2$ defined by

$$K = \Big\{(p,q) \in \Sigma^2 : p \in \Sigma, q \in C^{\gamma}(p) \cap \Sigma \text{ and } \mid y_p - y_q \mid \geq \gamma \cdot \beta\Big\}.$$

If β is small then K is a non empty compact set.

Define $H : \Sigma^2 \to [0, +\infty)$ by

$$H(p,q) = \mid \Pi^{\mathscr{F}}(p) - \Pi^{\mathscr{F}}(q) \mid .$$

As $H(p,q) = 0$ if only if $\Pi^{\mathscr{F}}(p) = \Pi^{\mathscr{F}}(q)$, then $H(p,q) > 0$ for all $(p,q) \in K$ because $\mid y_p - y_q \mid \geq \gamma \cdot \beta$. The continuity of $H(\cdot,\cdot)$ and the compactness of K imply that there exists a real number $\theta > 0$ such that

$$H(p,q) \geq \theta \tag{4.49}$$

for all $(p,q) \in K$.

From step 2 we obtain that $lenght(\Pi^{\mathscr{F}}(R^n(\zeta))) \to 0$ as $n \to \infty$.

Now, let us consider n be the natural number which satisfies

$$lenght(\Pi^{\mathscr{F}}(R^n(\zeta))) < \frac{\theta}{2} \tag{4.50}$$

and

$$lenght(R^n(\zeta)) > \beta. \tag{4.51}$$

Moreover we claim that there exists $p, q \in R^n(\zeta)$ such that $(p, q) \in K$. Indeed, let $p, q \in R^n(\zeta)$ such that the length of the curve $\overline{\zeta}$, parameterized in a such way that $D\overline{\zeta}(y) = (v(y), 1)$, is contained in $R^n(\zeta)$ between p and q is equal to β (see (4.51) given above).

In other hand as $\gamma < 1$ we have

$$\begin{aligned} lenght(\overline{\zeta}) &= \int_{y_q}^{y_p} \|D\overline{\zeta}(y)\| dy \\ &\leq \mid y_p - y_q \mid, \end{aligned}$$

and this implies that $\mid y_p - y_q \mid \geq \gamma \cdot \beta$, i,e., $(p, q) \in K$.

So using (4.49) we obtain that

$$H(p, q) \geq \theta. \tag{4.52}$$

Also notice that

$$H(p, q) \leq lenght(\Pi^{\mathscr{F}}(R^n(\zeta))). \tag{4.53}$$

Therefore using (4.50), (4.52) and (4.53) we obtain

$$\theta \leq H(p, q) \leq lenght(\Pi^{\mathscr{F}}(R^n(\zeta))) < \frac{\theta}{2}$$

and this is a contradiction. The proof of (4.48) of the step 3 follows.

Step 4. $\Pi^{\mathscr{F}}(R^n(\zeta))$ accumulate to 1 . Indeed, suppose that $\Pi^{\mathscr{F}}(R^n(\zeta))$ not accumulate to 1, then there exists $\tilde{\theta} > 0$ such that $\Pi^{\mathscr{F}}(R^n(\zeta)) \subset [0, 1 - \tilde{\theta}]$ for all $n \in \mathbb{N}$. Therefore as the foliation $\mathscr{F}$ is C^0 we can choose $\theta > 0$ such that $R^n(\zeta) \cap [0, 1] \times (1 - \theta, 1] = \emptyset$. Applying Corollary 4.2.12 after reparametrization of the curve ζ we obtain that there are $C > 0$ and

$\lambda > 1$ such that for all $t \in [0,1]$, we have

$$\begin{aligned} lengh(R^n(\zeta)) &= \int_0^1 \| \frac{dR^n(\zeta(t))}{dt} \| \, dt \\ &= \int_0^1 \| DR^n(\zeta(t)) \cdot \frac{d\zeta(t)}{dt} \| \, dt \\ &\geq C \cdot \lambda^n \cdot \int_0^1 \| \frac{d\zeta(t)}{dt} \| \, dt \\ &= C \cdot \lambda^n . lenght(\zeta). \end{aligned}$$

Then $lengh(R^n(\zeta)) \to \infty$ as $n \to \infty$ in contradiction with (4.48) of the step 3. Therefore, $\Pi^{\mathscr{F}}(R^n(\zeta))$ accumulate to 1.

Next, we argue in order to arrive a contradiction. Let us consider $0 < \eta < \delta_4$ and an integer n_0 in a such way that $\forall n \geq n_0$ $lengh(R^n(\zeta)) < \delta_4 - \eta$. So, if for $n \geq n_0$ and $R^n(\zeta) \cap [0,1] \times (1-\eta, 1) \neq \emptyset$ then $R^n(\zeta) \subset [0,1] \times (1-\delta_4, 1)$.

As $\Pi^{\mathscr{F}}([R^n(\zeta)])$ accumulate to 1, there is a sequence n_k such that $R^{n_k}(\zeta) \subset [0,1] \times (1-\delta_4, 1)$. We can apply (4.46) of Proposition 4.2.11, after reparametrization, to obtain that

$$lengh(R^{n_k}(\zeta)) \geq \lambda_2^{n_k - n_0} \cdot lengh(R^{n_0}(\zeta)).$$

As $n_k \to \infty$ we have that

$$lengh(R^{n_k}(\zeta)) \to \infty$$

and so we get a contradiction with (4.48) of the sep 3. The proof follows. ∎

Corollary 4.3.2.. *Let us consider $\mathscr{V}$ the C^1-neighborhood given by Theorem 4.3.1. Then for all $R \in \mathscr{V}$ and for all curve ζ tangent to C^γ such that $\zeta \cap (\bigcap_{i=0}^{\infty} R^{-i}(\Sigma)) \neq \emptyset$, there exists $n = n(R,\zeta)$ such that $\Pi^{\mathscr{F}}(R^n(\zeta)) \supseteq [0,1]$, where $\Pi^{\mathscr{F}}$ is the projection along the invariant foliation $\mathscr{F}$.*

Proof. Fix $R \in \mathscr{V}$ and a curve ζ tangent to C^γ such that $\zeta \cap (\bigcap_{i=0}^{\infty} R^{-i}(\Sigma)) \neq \emptyset$. Then Theorem 4.3.1 implies that $\bigcap_{i=0}^{\infty} R^{-i}(\Sigma)$ don't contain the curve ζ. Therefore there are a curve $\bar{\zeta} \subseteq \zeta$ with $\bar{\zeta}(0) = p$ and $\bar{\zeta}(1) = q$ for some $p, q \in \Sigma$ with $p \in \zeta \cap (\bigcap_{i=0}^{\infty} R^{-i}(\Sigma)) \neq \emptyset$ and a integer $n_0 = n_0(\bar{\zeta})$ such that $R^{n_0}(q) \in [0,1] \times \{0, a, b, 1\}$. But $R(\{y = 1\}) \subset \{y = 0\}$

and $\{y = 0\}$ is preserved by R so there is $n > n_0$ such that $\Pi^{\mathscr{F}}(R^n(\zeta)) \supseteq [0, 1]$. Therefore, from this the proof follows. ∎

Proof of Theorem 4.1.10. Fix R_0 as in Theorem 4.1.10. Take the neighborhoods $\mathscr{V}_5$ and $\mathscr{V}$ given by Corollary 4.2.12 and Theorem 4.3.1, respectively. Define $\mathscr{U} = \mathscr{V}_5 \cap \mathscr{V}$. Now, fix $R \in \mathscr{U}$ and also fix the invariant foliation $\mathscr{F}$ given by hypothesis (because that R is in particular a triangular map).

Claim A: All stable leaf $L \in \mathscr{F}$ intersecting $\bigcap_{n\geq 0} R^{-n}(H_{0,a} \cup H_{b,1})$ is accumulate by hyperbolic periodic points of saddle type.

Indeed, take a stable leaf $L \in \mathscr{F}$ intersecting $\bigcap_{n\geq 0} R^{-n}(H_{0,a} \cup H_{b,1})$ and take a curve ζ tangent to C^γ intersecting L. It follows from Corollary 4.3.2 the existence of a curve $\bar{\zeta} \subseteq \zeta$ and $n(\bar{\zeta}) \in \mathbb{N}$ such that $R^i(\overline{\gamma}) \subseteq H_{0,a} \cup H_{b,1}$ $\forall 0 \leq i \leq n(\overline{\gamma}) - 1$ and $R^{n(\overline{\gamma})}(\bar{\gamma})$ meets all leaf in $\mathscr{F}$.

Let $H(\bar{\zeta})$ be the horizontal band in Σ consisting of saturating $\bar{\zeta}$ by the foliation $\mathscr{F}$. By the property of $\bar{\zeta}$ and $n = n(\bar{\zeta})$ above we have that $R^n(H(\bar{\zeta}))$ crosses $H(\bar{\zeta})$ in a hyperbolic way. Then, by standard index arguments (see [56]) there is a periodic point of R on $R^n(H(\bar{\zeta})) \cap H(\bar{\zeta})$. Such a periodic point is hyperbolic saddle by Corollary 4.2.12 and the fact that the foliation $\mathscr{F}$ is R-contracting (see Remark 4.2.8). By taking ζ close to L, the band $H(\bar{\zeta})$ remain close to L and then we have that such a point is close to L. This proves the claim A.

Claim B: The hyperbolic periodic points of saddle type of R are dense in $\bigcap_{n\in\mathbb{Z}} R^n(H_{0,a} \cup H_{b,1})$. Indeed, take a point $z \in \bigcap_{n\in\mathbb{Z}} R^n(H_{0,a} \cup H_{b,1})$ and take a neighborhood V of z. We can choice a large enough integer n such that a neighborhood U of the leaf that contains $R^{-n}(z)$, i.e. a small horizontal band around the leaf, be applied into V. By Claim A there exists a periodic point of saddle type in U. Therefore the claim B follows.

To finish the proof of the transitivity of R (i.e., the invariant maximal set given by (4.3) is transitive) we will use the classical Birkhoff's criterium to prove transitivity: for all $p, q \in \bigcap_{n\in\mathbb{Z}} R^n(H_{0,a} \cup H_{b,1})$ and $\varepsilon > 0$ there are $z \in \bigcap_{n\in\mathbb{Z}} R^n(H_{0,a} \cup H_{b,1})$ and $n_z \in \mathbb{N}$ such that $d(z, p) < \varepsilon$ and $d(R^{n_z}(z), q) < \varepsilon$. Indeed, fix p, q and ϵ. By the above claim B we can assume that p and q are hyperbolic periodic points of saddle type. Fix a curve γ in $W^u(p)$ such that intersects transversally the stable manifold of q in some point z^*. Since the positive (resp. negative) orbit of z^* is asymptotic to q (resp. p) we have $z^* \in \bigcap_{n\in\mathbb{Z}} R^n(H_{0,a} \cup H_{b,1})$. By taking the negative orbit of z^* we have some $n_1^* \in \mathbb{N}$ such that

$$d(R^{-n_1^*}(z^*), p) < \varepsilon.$$

By taking the positive orbit of z^* we have some $n_2^* \in \mathbb{N}$ such that

$$d(R^{n_2^*}(z^*), q) < \varepsilon.$$

Then $z = R^{-n_1^*}(z^*)$ and $n_z = n_1^* + n_2^*$ works.

So finish the proof of Theorem 4.1.10. ■

Chapter 5

Singular cycles and C^k-robust transitive sets on manifolds with boundary: The continuous foliation case

In this chapter we improve the result obtain in the Chapter 3 making use of triangular maps given in Chapter 4. For this we only required sufficient differentiability to obtain C^1-linearizing coordinates. With those the invariant foliation $\mathscr{F}$ is only continuos and consequently the leaf application is only continuos. So, we can not apply the one-dimensional approach. Nevertheless, the Poncaré map is a triangular map according the definition in Chapter 4. Even more we expect that this approach work for the C^1 case.

5.1. Main Theorem

Theorem 5.1.1. *If $X \in \mathscr{X}^\infty(M, \partial M)$ exhibits a generic singular cycle associated to a singularity $\sigma \in \partial M$ with real eigenvalues $\lambda_{ss}, \lambda_s, \lambda_u$ satisfying*

$$\lambda_{ss} < \lambda_s < 0 < \lambda_u \;\; \textit{and} \;\; \lambda_s - \lambda_{ss} - 2\lambda_u > 0,$$

then for $k \geq 2$ large enough (lower that k in the Theorem 3.1.1), X also exhibits a C^k-robust transitive set containing σ.

So, an application of this result is the following corollary:

Corollary 5.1.2. *For every compact 3-manifold with nonempty boundary there is $X \in \mathscr{X}^\infty(M, \partial M)$ exhibiting a C^k-robust transitive set (for k large enough) which is not singular-hyperbolic for X neither $-X$.*

5.2. Proof of the Main Theorem

Here we proof the Theorem 5.1.1.

Let $X \in \mathscr{X}^\infty(M, \partial M)$ be exhibiting a generic singular cycle

$$\Gamma = \{\sigma, O, \gamma_0, \gamma_1\}$$

associated to a singularity $\sigma \in \partial M$.

Under generic hypothesis we can assume the existence of C^3 linearizing coordinates in a neighborhood of the singularity σ as well for the Poincaré map associated to the periodic orbit O. Moreover, there is a k large enough (with k smaller that k in Theorem 3.1.1), depending only on the eigenvalues only of σ and O, such that for any C^k vector field Y C^k-close to X, there are C^1-linearizing coordinates in the neighborhood of the continuations of the singularity σ as well for the Poincaré map associate to the continuation of the periodic point O, which depend smoothly on Y. From now on we fix such k.

The proof follows the same steps as in the proof in Theorem 3.1.1. We only need make some changes in the third step, that become from the fact that the foliation is not differentiable.

First of all we observe that there is a isolating block U for the cycle Γ.

We will prove that the set

$$\Lambda = \bigcap_{t \in \mathbb{R}} X_t(U)$$

is a C^k-robust transitive set of X, that is for all Y C^k-close to X, $\Lambda_Y = \bigcap_{t\in\mathbb{R}} Y_t(U)$ is a transitive set. Note that Theorem 5.1.1 follows directly from this since $\sigma \in \Gamma \subset \Lambda$.

To prove that Λ is a C^k-robust transitive set for X we need to do an accurate description for Poincaré maps involved in the dynamic around the cycle.

Fix a point $p \in O$ and a cross section S at p. Under generic hypothesis we can assume the existence of $\psi_X : S \to \Sigma$ a C^3 change coordinate on S that linearize the Poincaré map defined on S, where $\Sigma = [0, 1+\delta] \times [0, 1+\delta]$ for some $\delta > 0$. From now on we identify S with Σ trough ψ_X.

Let consider the Poincaré map $R : H_{0,a} \cup H_{b,\varphi} \subset \Sigma \to \Sigma$ defined by X on a neighborhood associate to the cycle Γ, where $H_{0,a}$ and $H_{b,\varphi}$ are horizontal strip as defined in Chapter 3, for some fixed constants a and b with $0 < a < b < \varphi(x)$ for all $x \in [0, 1+\delta]$, where φ is the C^3 map defined by the last connected component of the intersection between Σ and $W^s(\sigma)$. Moreover we can suppose that $\varphi(0) = 1$.

Let consider the Poincaré map Π associated to the periodic orbit O. By linearizing hypothesis we have $\Pi : H_{0,a} \subset \Sigma \to \Sigma$ is the linear map

$$(x, y) \to (\lambda \cdot x, \rho \cdot y);$$

where $0 < \lambda < 1$ and $\rho > 1$ are the eigenvalues for the periodic orbit O. Note that $R = \Pi$ on $H_{0,a}$.

By condition on the eigenvalues of the singularity σ, there exists a contracting C^3 R-invariant foliation $\mathscr{F}$ on Σ (see [6], [71]). Using that $R(\{y = a\}) \subset [0, 1+\delta] \times (1, 1+\delta)$ and $R(\{y = b\}) \subset [0, 1+\delta] \times (1, 1+\delta)$, this construction can be made in a such way that the set $\{y = 0\}$, $\{y = a\}$, $\{y = b\}$ and $\{y = \varphi(x)\}$ are leaves of this foliation. Moreover if $\Pi^{\mathscr{F}}$ denote the homology map, then its satisfies the properties:

$$\left|\frac{\partial \Pi^{\mathscr{F}}}{\partial x}(x, y)\right| < \frac{1}{2} \text{ and } \left|\frac{\partial \Pi^{\mathscr{F}}}{\partial y}(x, y)\right| > \frac{3}{4}.$$

Replacing q_0 by some positive iterate of it, say n and replacing q_1 by some negative iterate of it, say the same $-n$ and normalizing by the change coordinates $(\lambda^{-n}x, \rho^n y)$, we obtain a new Poincaré map in a such way that the invariant foliation satisfy:

$$\left|\frac{\partial \Pi^{\mathscr{F}}}{\partial x}(x,y)\right| \approx 0 \text{ and } \left|\frac{\partial \Pi^{\mathscr{F}}}{\partial y}(x,y)\right| \approx 1.$$

We can use this foliation to define new coordinates $(\bar{x},\bar{y}) = \psi(x,y)$ on Σ close to identity map, still linearizing, given by

$$(\bar{x},\bar{y}) = \psi(x,y) = (x, \Pi^{\mathscr{F}}(x,y)). \tag{5.1}$$

So, the map $\bar{R} = \psi \circ R \circ \psi^{-1}$ is defined for all $(\bar{x},\bar{y}) \in H_{0,a} \cup H_{b,1}$, and satisfies

$$\bar{R}(\bar{x},\bar{y}) = (\bar{f}(\bar{x},\bar{y}), \bar{g}(\bar{y})),$$

for some C^3 maps $\bar{f}(\cdot\,,\cdot)$ and $\bar{g}(\cdot)$. Note that $\bar{g}(a)$ and $\bar{g}(b)$ are greater than 1. Moreover, by assumption we have that $\bar{g}$ is strictly decreasing on $[b,1]$ and $\bar{g}(1) = 0$.

Let consider the $\bar{R}$ invariant *cone field* $C^\gamma : \bar{p} \mapsto C^\gamma(\bar{p}) \subset \mathbb{R}^2$, where

$$C^\gamma(\bar{p}) = \{\bar{v} \in \mathbb{R}^2 : \bar{v} = (\bar{u},\bar{w}); |\bar{u}| \leq \gamma \cdot |\bar{w}|\}.$$

The next proposition resume the most important properties of map $\bar{R}$ associate to generic cycles as in Theorem 5.1.1. Here $\|\cdot\|$ denote the maximum norm in $\mathbb{R}^2$.

Proposition 5.2.1. *Let $X \in \mathscr{X}^\infty(M,\partial M)$ be exhibiting a generic singular cycle as in Theorem 5.1.1. Then, shrinking the cross-section Σ if necessary, there is C^3-linearizing coordinated on Σ such that the Poincaré map $\bar{R} = (\bar{f},\bar{g})$ satisfy the following properties:*

a) $\bar{R}(\bar{x},\bar{y}) = (\lambda \cdot \bar{x}, \rho \cdot \bar{y})$ *for all* $(\bar{x},\bar{y}) \in H_{0,a}$.

b) $\bar{g}$ *has negative Schwarzian derivative on* $[b,1)$.

c) For $\gamma < 1$ *the vertical cone field* C^γ *is a* $\bar{R}$*-invariant cone field.*

d) There is positive constant K_0 *such that*

$$\bar{g}(\bar{y}) = \bar{y}_{\bar{R}(\bar{p})} \leq K_0|\bar{y}-1|^\alpha, \text{ for all } \bar{p} = (\bar{x},\bar{y}) \in H_{b,1}.$$

e) There is positive constant K_1 such that for all $\bar{p} = (\bar{x}, \bar{y}) \in H_{b,1}$ and $\bar{v} \in C^\gamma(\bar{p})$,

$$\|D\bar{R}(\bar{p}) \cdot \bar{v}\| \geq K_1 \cdot |\bar{y} - 1|^{\alpha - 1} \cdot \|\bar{v}\|.$$

f) For all $\bar{p} \in H_{0,a}$ and $\bar{v} \in C^\gamma(\bar{p})$ it have that

$$\|D\bar{R}(\bar{p}) \cdot \bar{v}\| = \rho \cdot \|\bar{v}\|.$$

Proof. Items a), c) and f) are immediate. The other items are consequences of Proposition 3.3.1. ■

To study the maximal invariant set Λ_Y for Y C^k-close to X we will need to introduce the respective Poincaré map associated to Y.

Let $O(Y)$, $\sigma(Y)$, φ_Y, $q_0(Y)$, $q_1(Y)$, $\gamma_0(Y)$ and $\gamma_1(Y)$ be the continuations of O, σ, φ, q_0, q_1, γ_0 and γ_1 respectively. Observe that the cross-section S remain transverse to any flow C^1-close to X. Take $\psi_Y : S \to \Sigma$ a C^1 change coordinate on S that linearize the Poincaré map induced by the Y-flow on S, that exist for all vector field Y C^k close to X. Identifying S with Σ trough ψ_Y, we can suppose that $q_0(Y) = (1, 0)$ and $q_1(Y) = (0, 1)$. Moreover let λ_Y and ρ_Y the eigenvalues associated to $O(Y)$, the continuation of λ and ρ respectively. Now let consider the uniform linearizing coordinates (x_1, x_2, x_3) in a neighborhood of the singularity $\sigma(Y)$ of Y, that exist for all vector field Y C^k close to X. Observe that $\Sigma_{in} = \{x_3 = 1\}$ and $\Sigma_{out} = \{x_2 = 1\}$ are transversal cross-sections to flow Y. Let $\lambda_{ss}(Y) < \lambda_s(Y) < 0 < \lambda_u(Y)$ be the eigenvalues of $DY(\sigma(Y))$, and denote $\alpha_Y = -\frac{\lambda_s(Y)}{\lambda_u(Y)}$ and $\beta_Y = -\frac{\lambda_{ss}(Y)}{\lambda_u(Y)}$.

The maps $\Pi_X, \Pi_{in,X}, \Pi_{loc,X}, \Pi_{out,X}$ as well as R_X have continuations $\Pi_Y, \Pi_{in,Y}$, $\Pi_{loc,Y}$, $\Pi_{out,Y}$ and R_Y for all vector field Y close to X. Note that the Poincaré first return map R_Y is given by

$$R_Y : H_{0,a} \cup H_{b,\varphi_Y} \subset \Sigma \to \Sigma$$

defined as Π_Y in $H_{0,a}$ and $\Pi_{out,Y} \circ \Pi_{loc,Y} \circ \Pi_{in,Y}$ in H_{b,φ_Y}. Here $H_{0,a} = [0, 1+\delta] \times [0, a]$ and $H_{b,\varphi_Y} = [0, 1+\delta] \times \{(x, y) : 0 \leq x \leq 1+\delta, y = \varphi_Y(x)\}$. Moreover we can suppose that $\Pi_Y(\{y = a\}) \subset [0, 1+\delta] \times \{(x, y) : 0 \leq x \leq 1+\delta, \varphi_Y(x) < y < 1+\delta\}$ and $R_Y\{y = b\}) \subset [-\delta, 1+\delta] \times \{(x, y) : 0 \leq x \leq 1+\delta, \varphi_Y(x) < y < 1+\delta\}$.

We have that

$$\bigcap_{t \in \mathbb{R}} Y_t(U) = Cl\left(\bigcup_{t \in \mathbb{R}} Y_t\left(\bigcap_{n \in \mathbb{Z}} R_Y^n(H_{0,a} \cup H_{b,\varphi_Y})\right)\right),$$

where $Cl(\cdot)$ denotes the closure operator. So, in order to prove that $\cap_{t\in\mathbb{R}} Y_t(U)$ is a transitive set, we only need to prove that the maximal invariant set

$$\bigcap_{n\in\mathbb{Z}} R_Y^n(H_{0,a}\cup H_{b,\varphi_Y}) \tag{5.2}$$

is a transitive set for R_Y.

To catch this set, let consider the C^1 coordinates $(\bar{x},\bar{y})$ given by

$$(\bar{x},\bar{y}) = \phi_Y(x,y) = \left(x, \Pi^{\mathscr{F}}(x,y) + b(\Pi^{\mathscr{F}}(x,y))\cdot\left[\frac{\Pi^{\mathscr{F}}(x,y)}{\Pi^{\mathscr{F}}(x,\varphi_Y(x))} - \Pi^{\mathscr{F}}(x,y)\right]\right),$$

where $b(\cdot)$ is the bump function such that $b(y)=0$ if $y\in[0,b]$ and $b(y)=1$ if $y\in[c,1+\delta]$ for some constant c, $b<c<1$, and $\mathscr{F}$ is the R invariant foliation.

We must remember that φ_Y depends smoothly on Y, so ϕ_Y. Moreover as $\frac{\Pi^{\mathscr{F}}(x,y)}{\Pi^{\mathscr{F}}(x,\varphi(x))} - \Pi^{\mathscr{F}}(x,y) = 0$ then for all vector field Y C^1 close to X we have that $b(\Pi^{\mathscr{F}}(x,y))\cdot\left[\frac{\Pi^{\mathscr{F}}(x,y)}{\Pi^{\mathscr{F}}(x,\varphi_Y(x))} - \Pi^{\mathscr{F}}(x,y)\right]$ is small in the C^1-norm because that $b(\cdot)$ only depends on X, so ϕ_Y is close to identity map. Furthermore $\phi_Y(\{y=\varphi_Y(x)\}) = \{\bar{y}=1\}$, $\phi_Y(\{y=a\}) = \{\bar{y}=a\}$ and $\phi_Y(\{y=b\}) = \{\bar{y}=b\}$.

In this coordinates let consider the map $\bar{R}_Y : H_{0,a}\cup H_{b,1}\subset\Sigma\to\Sigma$ given by

$$\bar{R}_Y(\bar{x},\bar{y}) = \phi_Y\circ R_Y\circ\phi_Y^{-1}. \tag{5.3}$$

So, for $(\bar{x},\bar{y})\in H_{0,a}\cup H_{b,1}$,

$$\bar{R}_Y(\bar{x},\bar{y}) = (\bar{f}_Y(\bar{x},\bar{y}), \bar{g}_Y(\bar{x},\bar{y})), \tag{5.4}$$

for some C^1 maps $\bar{f}_Y(\cdot,\cdot)$ and $\bar{g}_Y(\cdot,\cdot)$.

The next proposition resume the most important properties of map $\bar{R}_Y$ for all Y C^k close to X.

Proposition 5.2.2. *Let $X\in\mathscr{X}^\infty(M,\partial M)$ be exhibiting a generic singular cycle as in Theorem 5.1.1. Then, shrinking the cross-section Σ if necessary, there are fourth number K_0, K_1, ν and μ with $K_0, K_1 > 0$ and $1<\nu\leq\mu$ such that for each $0<\gamma<1$ and for any Y C^k-close to X, there are a C^1-coordinates on Σ such that the Poincaré map $\bar{R}_Y = (\bar{f}_Y, \bar{g}_Y)$ satisfy the following properties:*

a)

$$\nu \cdot \mu^{\frac{1-\alpha_Y}{\alpha_Y}} > 1.$$

b) $\bar{R}_Y$ *is* C^1 *close to* $\bar{R}$.

c) There exists a C^0 $\bar{R}_Y$*-invariant contracting foliation* $\bar{\mathscr{F}}$ *depending smoothly on* Y*.*

d) There exists a cone field C^γ *which is transversal to* $\bar{\mathscr{F}}$*.*

e) For all $\bar{p} \in H_{0,a}$ *and* $\bar{v} \in C^\gamma(\bar{p})$ *it have that*

$$\nu \cdot \|\bar{v}\| \leq \|D\bar{R}_Y(\bar{p}) \cdot v\| \leq \mu \cdot \|\bar{v}\|.$$

f) K_0 *satisfies:*

$$\bar{g}_Y(x, y) \leq K_0|y-1|^{\alpha_Y}, \text{ for all } \bar{p} = (\bar{x}, \bar{y}) \in H_{b,1}.$$

g) K_1 *satisfies: for all* $\bar{p} = (\bar{x}, \bar{y}) \in H_{\bar{b},1}$ *and* $\bar{v} \in C^\gamma(\bar{p})$*,*

$$\|D\bar{R}_Y(\bar{p}) \cdot \bar{v}\| \geq K_1 \cdot |\bar{y}-1|^{\alpha_Y - 1} \cdot \|\bar{v}\|.$$

Proof. Fix $X \in \mathscr{X}^\infty(M, \partial M)$ be exhibiting a generic singular cycle as in Theorem 5.1.1.

Fix $\gamma < 1$. Let consider Y C^k close to X.

Define

$$\nu = \rho - \varepsilon \text{ and } \mu = \rho + \varepsilon$$

for some positive ε. So, for all Y C^k close to X,

$$\nu\mu^{\frac{1-\alpha_Y}{\alpha_Y}} = (\rho - \epsilon)(\rho + \epsilon)^{\frac{1-\alpha_Y}{\alpha_Y}} \to \rho^{\frac{1}{\alpha_Y}} > 1 \qquad \text{as} \qquad \epsilon \to 0^+.$$

Then, a) holds for ε close to 0.

Note that $\bar{R}_Y$ is C^1 close to $\bar{R}$ on $H_{0,a}$, because R_Y is lineal and close to R on $H_{0,a}$, and ϕ_Y is close to identity map. Now we prove that $\bar{R}_Y$ is C^1 close to $\bar{R}$ on $H_{b,1}$.

First of all we will obtain the expression of $\bar{R}_Y(\bar{p})$ for all $\bar{p} \in H_{b,1}$.

Let consider $\Pi_{in,Y}$, $\Pi_{out,Y}$ and $\Pi_{loc,Y}$ the continuation of Π_{in}, Π_{out} and Π_{loc}, respectively.

Denote by $\bar{\Pi}_{in,Y} = \Pi_{in,Y} \circ \phi_Y^{-1}$ and $\bar{\Pi}_{out,Y} = \phi_Y \circ \Pi_{out,Y}$. Let consider $\bar{q}_1(Y) = (0, 1)$.

By Taylor formula,

$$\bar{\Pi}_{in,Y}(\bar{x},\bar{y}) = \bar{\Pi}_{in,Y}(\bar{q}_1(Y)) + D\bar{\Pi}_{in,Y}(\bar{q}_1(Y)) \cdot (\bar{x},\bar{y}-1) + \bar{\Theta}_{in,Y}(\bar{x},\bar{y}-1),$$

where

$$D\bar{\Pi}_{in,Y}(\bar{q}_1(Y)) = \begin{bmatrix} \bar{a}_Y & \bar{b}_Y \\ \bar{c}_Y & \bar{d}_Y \end{bmatrix}$$

with $\bar{a}_Y, \bar{b}_Y, \bar{c}_Y, \bar{d}_Y \in \mathbb{R}$ and $\text{lím}_{(\bar{x},\bar{y})\to(0,1)} \frac{\bar{\Theta}_{in,Y}(\bar{x},\bar{y}-1)}{\|(\bar{x},\bar{y}-1)\|} = 0$.

As $D\bar{\Pi}_{in,Y}(\bar{q}_1(Y)) \cdot e_1 = (\bar{a}_Y, \bar{c}_Y)$ and $D\bar{\Pi}_{in,Y}(\bar{q}_1(Y)) \cdot e_2 = (\bar{b}_Y, \bar{d}_Y)$, by assumption we have that $\bar{a}_Y$ and $\bar{d}_Y$ are negatives and $\bar{c}_Y = 0$. Also we have that $\bar{\Pi}_{in,Y}(\bar{q}_1(Y)) = (x_0(Y), 0)$. Then

$$\begin{aligned} \bar{\Pi}_{in,Y}(\bar{x},\bar{y}) &= (x_0(Y) + \bar{a}_Y \cdot \bar{x} + \bar{b}_Y \cdot (\bar{y}-1) + \bar{\Theta}_{1,Y}(\bar{x},\bar{y}-1), \\ &\quad \bar{d}_Y \cdot (\bar{y}-1) + \bar{\Theta}_{2,Y}(\bar{x},\bar{y}-1)) \end{aligned}$$

where $\bar{\Theta}_{in,Y} = (\bar{\Theta}_{1,Y}, \bar{\Theta}_{2,Y})$.

Note that $\bar{\Theta}_{2,Y}(\bar{x},0) = 0$.

Also, by Taylor formula, calling $(\tilde{x}_1, \tilde{x}_3)$ the coordinates on $\hat{\Sigma}_{out}$, we get

$$\bar{\Pi}_{out,Y}(\tilde{x}_1,\tilde{x}_3) = \bar{\Pi}_{out,Y}(0,0) + D\bar{\Pi}_{out,Y}(0,0) \cdot (\tilde{x}_1,\tilde{x}_3) + \bar{\Theta}_{out,Y}(\tilde{x}_1,\tilde{x}_3),$$

where

$$D\bar{\Pi}_{out,Y}(0,0) = \begin{bmatrix} \hat{a}_Y & \hat{b}_Y \\ \hat{c}_Y & \hat{d}_Y \end{bmatrix}$$

with $\hat{a}_Y, \hat{b}_Y, \hat{c}_Y, \hat{d}_Y \in \mathbb{R}$ and $\text{lím}_{(\tilde{x}_1,\tilde{x}_3)\to(0,0)} \frac{\bar{\Theta}_{out,Y}(\tilde{x}_1,\tilde{x}_3)}{\|(\tilde{x}_1,\tilde{x}_3)\|} = 0$.

As $D\bar{\Pi}_{out,Y}(0,0) \cdot e_1 = (\hat{a}_Y, \hat{c}_Y)$ and $D\bar{\Pi}_{out,Y}(0,0) \cdot e_2 = (\hat{b}_Y, \hat{d}_Y)$, by assumption we have that $\hat{a}_Y$ and $\hat{d}_Y$ are positives and $\hat{c}_Y = 0$. Moreover we supposed that $\bar{\Pi}_{out,Y}(0,0) = (1,0)$.

Then

$$\begin{aligned} \bar{\Pi}_{out,Y}(\tilde{x}_1,\tilde{x}_3) &= (1 + \hat{a}_Y \cdot \tilde{x}_1 + \hat{b}_Y \cdot \tilde{x}_3 + \bar{\Theta}_{3,Y}(\tilde{x}_1,\tilde{x}_3), \\ &\quad \hat{d}_Y \cdot \tilde{x}_3 + \bar{\Theta}_{4,Y}(\tilde{x}_1,\tilde{x}_3)) \end{aligned}$$

where $\bar{\Theta}_{out,Y} = (\bar{\Theta}_{3,Y}, \bar{\Theta}_{4,Y})$.

Note that $\bar{\Theta}_{4,Y}(\tilde{x}_1, 0) = 0$.

On the other side, remember that

$$\bar{\Pi}_{loc,Y}(x_1, x_2) = (x_1 \cdot x_2^{\beta}, x_2^{\alpha}).$$

Putting $A = \Big[x_0(Y) + \bar{a}_Y \cdot \bar{x} + \bar{b}_Y \cdot (\bar{y} - 1) + \bar{\Theta}_{1,Y}(\bar{x}, \bar{y} - 1)\Big]$ and $B = \Big[\bar{d}_Y \cdot (\bar{y} - 1) + \Theta_{2,Y}(\bar{x}, \bar{y} - 1)\Big]^{\beta_Y}$ we denote by

$$\tilde{x}_1 = A \cdot B$$

and

$$\tilde{x}_3 = \Big[\bar{d}_Y \cdot (\bar{y} - 1) + \Theta_{2,Y}(\bar{x}, \bar{y} - 1)\Big]^{\alpha_Y}.$$

As $\bar{R}_Y = (\bar{f}_Y(\bar{x}, \bar{y}), \bar{g}_Y(\bar{x}, \bar{y})) = \bar{\Pi}_{out,Y} \circ \bar{\Pi}_{loc,Y} \circ \bar{\Pi}_{in,Y}$ (in $H_{b,1}$) we get that

$$\bar{f}_Y(\bar{x}, \bar{y}) = 1 + \hat{a}_Y \cdot \tilde{x}_1 + \hat{b}_Y \cdot \tilde{x}_3 + \bar{\Theta}_{3,Y}(\tilde{x}_1, \tilde{x}_3) \tag{5.5}$$

and

$$\begin{aligned} \bar{g}_Y(\bar{x}, \bar{y}) &= \hat{d}_Y \cdot \Big[\bar{d}_Y \cdot (\bar{y} - 1) + \bar{\Theta}_{2,Y}(\bar{x}, \bar{y} - 1)\Big]^{\alpha_Y} + \\ &\quad \bar{\Theta}_{4,Y}(\tilde{x}_1, \tilde{x}_3). \end{aligned} \tag{5.6}$$

Item b) is consequence of (5.3), (5.4), (5.6) (5.5) and the fact that $\phi_X(\cdot) = \phi(\cdot)$ and $\phi_Y(\cdot)$ is C^1 close to identity map.

Item c) can be constructed as in [6] (see too [71]) because that ψ_Y is C^1-close to identity map. Indeed the bounds for partial derivatives of $\bar{R}_Y = (\bar{f}_Y, \bar{g}_Y)$ come from the respective bounds for partial derivatives of $R_Y = (f_Y, g_Y)$. Using that $\bar{R}_Y(\{\bar{y} = b\}) \subset [0, 1+\delta] \times (1, 1+\delta)$ and $\bar{R}_Y(\{\bar{y} = a\}) \subset [0, 1+\delta] \times (1, 1+\delta)$, this construction can be made in a such way that the sets $\{\bar{y} = 0\}$, $\{\bar{y} = a\}$, $\{\bar{y} = b\}$ and $\{\bar{y} = 1\}$ are leaves of this foliation. Observe that the invariant foliation for $\bar{R}$ obtained by this construction is given by horizontal lines.

Item d) is consequence from item b), the continuous variation of invariant foliation and the fact that the existence of cone field is a open property in the C^1 topology.

Proof item e). From b) we have that $\bar{R}_Y$ is C^1 close to $\bar{R}$ on $H_{0,a}$. Take ε as in a) and Y close to X such that for all $\bar{p} \in H_{0,a}$ and for all $\bar{v} \in C^{\gamma}(\bar{p})$, $\| D\bar{R}_Y(\bar{p}) \cdot \bar{v} - D\bar{R}(\bar{p}) \cdot \bar{v} \| <$

$\varepsilon \cdot \parallel \bar{v} \parallel$. Moreover as $\parallel DR(\bar{p}) \cdot \bar{v} \parallel = \rho \parallel \bar{v} \parallel$, we get

$$\nu \cdot \|v\| \leq \|D\bar{R}_Y(\bar{p}) \cdot \bar{v}\| \leq \mu \cdot \|\bar{v}\|,$$

and this prove e).

Item f) is consequence (5.6).

Finally, to prove item g) note that for all $\bar{p} \in H_{b,1}$ and $\bar{v} = (\bar{u}, \bar{w})^T \in C^\gamma(\bar{p})$, $|\bar{w}| = \parallel \bar{v} \parallel$ because that $\gamma < 1$. Moreover, taken derivatives in (5.6) and using that $\bar{\psi}_Y(\cdot, \cdot)$ is C^1 close to the identity map, we get that there exists a constant $K_1 = K_1(X) > 0$ such that for all $\bar{p} \in H_{b,1}$ and $\bar{v} = (\bar{u}, \bar{w})^T \in C^\gamma(\bar{p})$,

$$\begin{aligned} \parallel D\bar{R}(\bar{p}) \cdot \bar{v} \parallel &\geq |\bar{w}| \cdot \text{máx}\Big\{ - \Big|\frac{\bar{u}}{\bar{w}}\Big| \cdot |\partial_{\bar{x}}\bar{g}_Y(\bar{x}, \bar{y})| + |\partial_{\bar{y}}\bar{g}_Y(\bar{x}, \bar{y})|\Big\} \\ &\geq |\bar{w}| \cdot \text{máx}\Big\{ - \gamma \cdot |\partial_{\bar{x}}\bar{g}_Y(\bar{x}, \bar{y})| + |\partial_{\bar{y}}\bar{g}_Y(\bar{x}, \bar{y})|\Big\} \\ &\geq K_1 \cdot |\bar{w}| \cdot |\bar{y} - 1|^{\alpha_Y - 1} \\ &= K_1 \cdot \parallel \bar{v} \parallel \cdot |\bar{y} - 1|^{\alpha_Y - 1}, \end{aligned}$$

and so g) follows.

So the proof follows. ■

Proof of the Theorem 5.1.1 (Main Theorem).

Indeed, using Proposition 5.2.1, Proposition 5.2.2 and Theorem 4.1.10 (see Chapter 4) we get that the maximal invariant set

$$\bigcap_{n \in \mathbb{Z}} \bar{R}_Y^n(H_{0,a} \cup H_{b,1})$$

is transitive.

So, the maximal invariant set given in (5.2) is transitive. The proof of the Main Theorem follows. ■

Chapter 6

An attracting singular-hiperbolic set containing a non-trivial repeller hyperbolic

The chain recurrent set of an Anosov three-dimensional vector field (e.g. [21]) is an example of a non-transitive singular-hyperbolic attracting set containing hyperbolic repellers. Observe that this example has no singularities. In this paper we construct a C^∞ three-dimensional vector field that exhibits a singular-hyperbolic attracting set containing a singularity which is not Lorenz-like and a non trivial repeller hyperbolic set.

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6.1. Statement of the Main Theorem

Newhouse in [57] proves that the non wandering set of any codimension-one Anosov diffeomorphism on a compact connected C^∞ manifold without boundary of a dimension greater than or equal to two is all the ambient manifold. Later, it was conjectured by Verjovsky [84] that a codimension-one Anosov flow on a compact manifold of a dimension greater than or equal to three has as its non wandering set, the whole manifold. Franks and Williams [21] has given a counterexample of a three-dimensional Anosov flow whose chain recurrent set is a non-transitive hyperbolic attracting set containing a non trivial repeller hyperbolic set but does not contain singularities.

This motivates the question if non-transitive singular-hyperbolic attracting sets exist containing both singularities and non trivial hyperbolic repellers. We give a positive answer for this question.

Notice that there are C^∞ vector fields on the unitary sphere $S^2 \subset \mathbb{R}^3$ which exhibit a non transitive hyperbolic attracting set containing a trivial repeller. The example provided in [49] show that there is a C^∞ vector field on a closed 3-manifold, whose nonwandering set is contained in the union of a repelling singularity and a singular-hyperbolic attracting set without Lorenz-like singularities and without non trivial repeller hyperbolic sets (see Theorem A). We observe that the geometric Lorenz attractor [24], Lorenz attractor [82] and singular Horsesoes [34] are examples of singular-hyperbolic attracting sets which do not contain non trivial repeller hyperbolic sets. However, we exhibit an example of a singular-hyperbolic attracting set containing a singularity which is not Lorenz-like and a non trivial repeller hyperbolic set.

Hereafter M denotes a compact 3-manifold. A C^r flow $\mathbb{X}_t$ on M is a C^r action $I\!R \times M \to M$, $r \geq 1$. We always assume that $\mathbb{X}_t$ is the integral solution of a C^r vector field denoted by $\mathbb{X}$. An *orbit* of $\mathbb{X}$ is the set $\{\mathbb{X}_t(p) : t \in I\!R\}$ for some $p \in M$. The *omega-limit set* of a point p is the set $w_{\mathbb{X}}(p) = \{x \in M : \exists \{t_n\},\ \text{lím}_{n\to\infty} t_n = \infty \text{ and } x = \text{lím}_{n\to\infty} \mathbb{X}_{t_n}(p)\}$. A *singularity* of $\mathbb{X}$ is a point $\sigma \in M$ such that $\mathbb{X}(\sigma) = 0$. A *periodic orbit* of $\mathbb{X}$ is an orbit such that $\mathbb{X}_T(p) = p$ for some $p \in M$ and minimal $T > 0$. A *closed orbit* of $\mathbb{X}$ is either a singularity or a periodic orbit of $\mathbb{X}$.

A compact set $\Lambda \subset M$ is: *Invariant* if $\mathbb{X}_t(\Lambda) = \Lambda$, $\forall t \in I\!R$, *Transitive* if $\Lambda = w_{\mathbb{X}}(p)$ for some $p \in \Lambda$, *Trivial* if Λ is a closed orbit of $\mathbb{X}$, *isolated* if there is a compact neighborhood U of Λ (U is called isolated block) such that $\Lambda = \bigcap_{t \in I\!R} \mathbb{X}_t(U)$. A *attracting set* of $\mathbb{X}$ is a compact invariant set of $\mathbb{X}$ which is isolated and has a positively invariant isolated block U, i.e., $\mathbb{X}_t(U) \subset U, \forall t \geq 0$. A *attractor* is a transitive attracting set. A *repelling set* of $\mathbb{X}$ is a attracting set for the time reversed vector field $-\mathbb{X}$ and a *repeller* of $\mathbb{X}$ is a transitive repelling set of $\mathbb{X}$.

A compact invariant set Λ de $\mathbb{X}$ is hyperbolic if there is a continuos invariant tangent bundle decomposition $T_\Lambda M = E^s_\Lambda \oplus E^{\mathbb{X}}_\Lambda \oplus E^u_\Lambda$and positive constants K, λ such that E^s_Λ, is contracting, i.e., $||D(\mathbb{X}_t)_p/E^s_p|| \leq Ke^{-\lambda t}$, $\forall t > 0, \forall p \in \Lambda$, E^u_Λ is expanding, i.e., $||D(\mathbb{X}_{-t})_p/E^u_p|| \leq Ke^{-\lambda t}$, $\forall t > 0, \forall p \in \Lambda$, $E^{\mathbb{X}}_\Lambda$ is the one-dimensional subbundle generated by $\mathbb{X}$ in Λ. If in addition $E^s_p \neq 0$ and $E^u_p \neq 0$ for all $p \in \Lambda$ then we say that Λ is saddle-type.

The Invariant Manifold Theory [31] asserts that if Λ is a hyperbolic set and $p \in \Lambda$, then the sets $W^{ss}_{\mathbb{X}}(p) = \{x \in M : \text{lím}_{t\to\infty} d(\mathbb{X}_t(x), \mathbb{X}_t(p)) = 0\}$ and $W^{uu}_{\mathbb{X}}(p) = W^{ss}_{-\mathbb{X}}(p)$ are immersed C^1 submanifolds of M. These manifolds are called *strong stable* and *strong unstable* manifolds of p. It turns out that $W^{ss}_{\mathbb{X}}(p)$ and $W^{uu}_{\mathbb{X}}(p)$ are respectively tangent to the linear spaces E^s_p and E^u_p at p. A singularity or a periodic orbit is *hyperbolic* (of saddle-type) if it as a compact invariante set.

Let Λ be a compact invariant set of $\mathbb{X}$. A *continuos invariant splitting* $T_\Lambda M = E_\Lambda \oplus F_\Lambda$ over Λ is *dominated* if there are positive constants K, λ such that $||D(\mathbb{X}_t)_p/E^s_p|| \cdot ||D(\mathbb{X}_{-t})/E^s_{\mathbb{X}_t(p)}|| \leq Ke^{-\lambda t}$, $\forall t \geq 0, \forall p \in \Lambda$. A compact invariant set Λ is *partially hyperbolic* if it exhibits a dominated splitting $T_\Lambda M = E^s_\Lambda \oplus E^{cu}_\Lambda$ such that E^s_Λ is contracting, i.e. $||D(\mathbb{X}_t)_p/E^s_p|| \leq Ke^{-\lambda t}$, $\forall t \geq 0, \forall p \in \Lambda$. A compact invariant set Λ of $\mathbb{X}$ is a *singular-hyperbolic set* if it has singularities (all hyperbolic) and is partially hyperbolic with volume expanding central direction E^{cu}_Λ, i.e. $|\det(D(\mathbb{X}_t)_p/E^{cu}_p)| \geq K^{-1}e^{\lambda t}$, $\forall t \geq 0, \forall p \in \Lambda$. A *singular-hyperbolic attractor* is a non-trivial attractor which is also a singular hyperbolic set. The most classical example of singular-hyperbolic attractors are the geometric Lorenz ones [24]. Other examples different from Lorenz's are the ones in [54] or [7].

Now we can state our result:

Theorem A. *There is a vector field C^∞ on a closed 3-manifold exhibiting a attracting singular-hyperbolic set which contains a singularity that is not Lorenz-like and a non trivial repeller hyperbolic set.*

The idea of the proof is as follows. We obtain a derived f from Anosov map $\mathcal{A}$ on T^2 somewhat similar to the derived from Anosov of Smale [76]. See also [87] and [20] with the difference that f has two sources $\mathcal{O}_1$ and $\mathcal{O}_2$. See Theorem 6.2.1. In the suspension of (T^2, f), we have taken open neighborhoods U_1, U_2 of periodic orbits respectively generated by $\mathcal{O}_1$ and $\mathcal{O}_2$. The result is accomplished by doing two surgeries, one on ∂U_1 as in [87] and one on ∂U_2 as in [49]. The organization of the paper is as follows. In Section 2 we give the proof of Theorem 6.2.1. In Section 3 we prove Theorem A.

6.2. Construction of the DA diffeomorphism

The following results is a modification of Theorem A given in [87]. We begin with a hyperbolic automorphism $g : T^2 \to T^2$ with the following properties: g is induced by a linear automorphism $A : \mathbb{R}^2 \to \mathbb{R}^2$ which can be represented by an integer matriz with determinant 1, with a real eigenvalue λ which satisfies $0 < \lambda < 1$ and the other eigenvalue μ greater than one in absolute value.

Theorem 6.2.1. *Corresponding to $g : T^2 \to T^2$ there is a diffeomorphism $f : T^2 \to T^2$ such that:*

(a) The wandering set $\Omega(f) = \{\mathcal{O}_1, \mathcal{O}_2\} \cup \Lambda$ where $\mathcal{O}_1$ and $\mathcal{O}_2$ are (points) sources and Λ is a one-dimensional attractor which contains three saddles-type fixed points;

(b) $\Omega(f)$ has a hyperbolic structure;

(c) For any $p \in \Lambda$, the stable space of p associated with f agrees with the stable space of p associated with g.

Proof. Choose a real valued C^∞ function $\rho(t)$ with graph as in figure 6.1. We summarize the properties that we wish ρ to have:

(a1) $\rho(t) = \rho(-t)$, for all $t \in \mathbb{R}$.

(a2) $\rho(t) = 0$ on a neighborhood of 0 and $\rho(t) = 0$, for all $t \geq 1$.

(a3) There exists $0 < \beta^* < 1$ such that $\rho(\beta^*) = 0$.

(a4) There exists $0 < a < \beta^* < b < 1$ such that $\rho(a) = \sup_{x\in[0,1]} \rho(x) > 1 - \lambda$ and $-\lambda < \rho(b) = \text{ínf}_{x\in[0,1]} \rho(x) < 0$.

(a5) $\int_0^a \rho(t)dt > (1 - \lambda)a$ and $\int_0^1 \rho(t)dt = 0$.

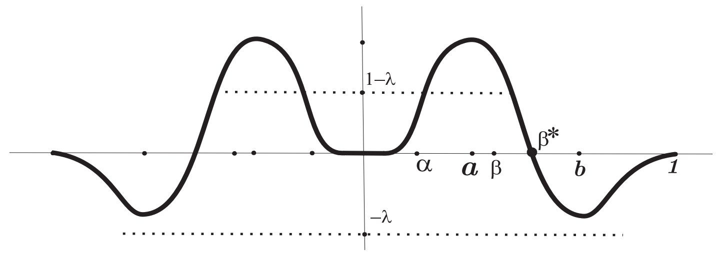

Figura 6.1: graph of ρ

Now define $\psi : \mathbb{R} \to \mathbb{R}$ by $\psi(x) = \lambda x + \int_0^x \rho(t)dt$, for $x \in \mathbb{R}$. We claim that ψ holds the following properties:

(b1) $\psi(-x) = -\psi(x)$, for all $x \in \mathbb{R}$.

(b2) ψ is a diffeomorphism. From $(a1)$ and $(a4)$, $\rho(x) > -\lambda$ for all $x \in I\!R$. We then obtain $\psi'(x) > 0$ for all $x \in I\!R$.

(b3) There is $0 < \alpha < a$ such that $\psi(x) = \lambda x$ for all $0 < x < \alpha$. In fact, by $(a2)$, there exists $0 < \alpha < a$ such that $\rho(x) = 0$ for all $x \in (-\alpha, \alpha)$, which implies $\int_0^x \rho(t)dt = 0$. Using the definition of ψ it follows $(b3)$.

(b4) There is β with $a < \beta < \beta^*$ such that $\psi(x) > x$ for all $x \in (a, \beta)$. We take $\beta \in (a, \beta^*)$ such that $\rho(\beta) = 1 - \lambda$. From $(a4)$, $\rho(t) > 1 - \lambda$ for all $a < t < \beta$. Therefore, we obtain $\int_a^x \rho(t)dt > (1 - \lambda)x - (1 - \lambda)a$, for all $x \in (a, \beta)$. Let $a < x < \beta$. Using the definition of ψ, $(a5)$, and the inequality above we obtain $(b4)$.

(b5) $\psi(x) = \lambda x$ for all $|x| \geq 1$. Indeed, by $(a2)$ we have that $\rho(t) = 0$ for all $t \geq 1$. By $(a5)$, $\int_0^1 \rho(t)dt = 0$, then by definition of ψ we obtain the requerid property.

For $\varepsilon > 0$ small enough exists $b_1 = b_1(\varepsilon) \in (0, \alpha)$ and $b_2 = b_2(\varepsilon) \in (\beta, \beta^*)$ close to β such that $-\lambda < \rho(x) < 1 - \lambda - \varepsilon$ for all $|x| < b_1$ or $|x| > b_2$. We define $\overline{\lambda} = \sup\{\psi'(x) : |x| < b_1 \text{ or } |x| > b_2\}$.

(b6) $\lambda < \overline{\lambda} < 1$. In fact, for $\varepsilon > 0$ fixed, we have $0 < \lambda + \rho(x) < 1 - \varepsilon$, for all $|x| < b_1$ or $|x| > b_2$. Moreover, x^* exists with $|x^*| > b_2$ such that $\rho(x^*) > 0$. It follows that $\rho(x^*) + \lambda > \lambda$. Therefore, $\lambda < \overline{\lambda} \leq 1 - \varepsilon < 1$.

We can now construct a slight modification DA diffeomorphism.

Let $\theta : \mathbb{R} \to \mathbb{R}$ be a C^∞ bump function satisfying $0 \leq \theta(t) \leq 1$ for all $\in \mathbb{R}$, $\theta(t) \equiv 1$ on $[-b_2, b_2]$, $\theta(t) \equiv 0$ for $|t| \geq 1$.

Choose linear coordinates u_1, u_2 on the neighborhood U of 0 in T^2, such that u_1 is parallel to the eigendirection corresponding to λ and u_2 is parallel to the eigendirection corresponding to μ.

We suppose that $U = \{q \in T^2 : u_1^2(q) + u_2^2(q) < 2\}$. We now define $f : T^2 \to T^2$ by altering g on U as follows: For $q \in U$, let $u_1(f(q)) = \psi(u_1(q))\theta(r) + u_1(g(q))(1 - \theta(r))$ where $r = \sqrt{u_1^2(q) + u_2^2(q)}$, and $u_2(f(q)) = u_2(g(q))$. Note that f agrees with g outside $V_0 = \{q \in T^2 : u_1^2(q) + u_2^2(q) < 1\}$.

Since ψ satisfies (b1) to (b6), by construction f has two sources $\mathcal{O}_1, \mathcal{O}_2$ and three saddle-like fixed points.

We note that f leaves the lines invariant $u_2 = \textit{constant}$ on U so that in fact f leaves the foliation of stable manifolds of g invariant on T^2. Thus if E^s is the stable bundle associated to g, the bundle is invariant under Df. Notice that the unstable bundle E^u associated to g is not invariant under Df.

Consider $V = \{q \in U : b_1 < \sqrt{u_1^2(q) + u_2^2(q)} \leq b_2\}$, from the properties $(b1) - (b6)$, the property $\mid Df_x(v) \mid \leq \bar{\lambda} \mid v \mid$ for $x \in U - V$ and $v \in E^s_x$ follows. Therefore as is proven in [20], pag. 276-278, the set $\Lambda = \cap_{m \geq 0} f^m(T^2 - V)$ is an attractor hyperbolic set for f. Moreover, following the demostration of Theorem A of Williams [87], the set Λ satisfies $(a), (b)$ and (c) of the Theorem 6.2.1; and the result is concluded. ■

6.3. Proof of the Main Theorem

Here we given the prove of the Theorem A.

Proof. Using DA-diffeomorphism f on T^2 given by Theorem 6.2.1 we obtain the suspension flow of f on a $3-$manifold whose non wandering set is a singular-hyperbolic attracting set which contains a singularity (not Lorenz-like), an attractor and a repeller that are all hyperbolic.

Let $(N, \mathbb{Y}^1)$ be the suspension of (T^2, f). That is, $N = (T^2 \times [0,1])/\sim$ where the equivalence $\sim$ in $T^2 \times [0,1]$ given by $(x,0) \sim (y,1)$ if only if $f(x) = y$. The flow $\mathbb{Y}^0 \in \mathfrak{X}^r(T^2 \times [0,1])$ given by $\mathbb{Y}^0(p,t) = (1,0)$ is the pullback of the flow $(N, \mathbb{Y}^1)$ through the canonical projection.

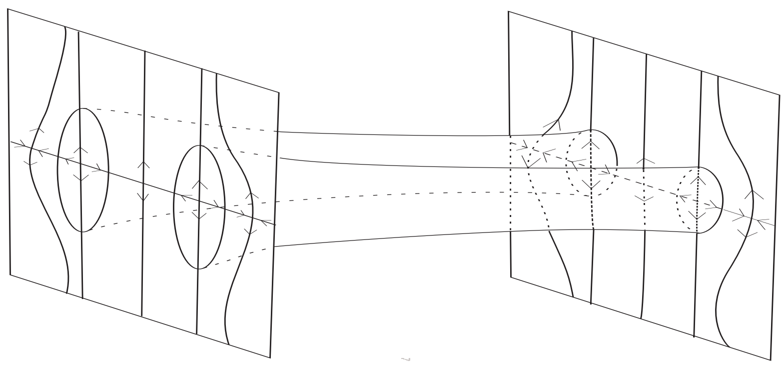

Figura 6.2: $(p,t) \in T^2 \times [0,1] \mapsto \mathbb{Y}^0(p,t) = (1,0)$

The non-wandering set of the vector field $\mathbb{Y}^1$ consists of a non trivial hyperbolic attractor A and two repeller periodic orbits $\mathcal{P}_\infty$ y $\mathcal{P}_\in$ associated with the fixed points $\mathcal{O}_\infty$ and $\mathcal{O}_\in$ of the DA-diffeomorphism given by the Theorem 6.2.1, that is,

$$\Omega(\mathbb{Y}^1) = A \cup \{\mathcal{P}_\infty, \mathcal{P}_\in\}.$$

Choosing the neighborhoods U_l of $\mathcal{P}_\updownarrow$, for $l = 1, 2$ in N homeomorphas to solids torus such that the stable foliation of A intersects with the toruses ∂U_1 and ∂U_2 in two of Reeb's components. See Figure 6.3.

In $U_1 \cup U_2$ we carry out a surgery in order to place instead of U_1 a non trivial repeller R similar to the Franks-Williams's [21] construction and replacing U_2 by a neighborhood with two singularities, a hyperbolic saddle that is not Lorenz-like σ_2 and a repeller singularity σ_1 the same as Morales [49]. See figure (6.4).

In this way, the three dimensional closed manifold M and vector field $\mathbb{X}$ on M of class C^∞ is obtained satisfying

$$\Omega(\mathbb{X}) = \{\sigma_1\} \cup A \cup R \cup W^u(\sigma_2),$$

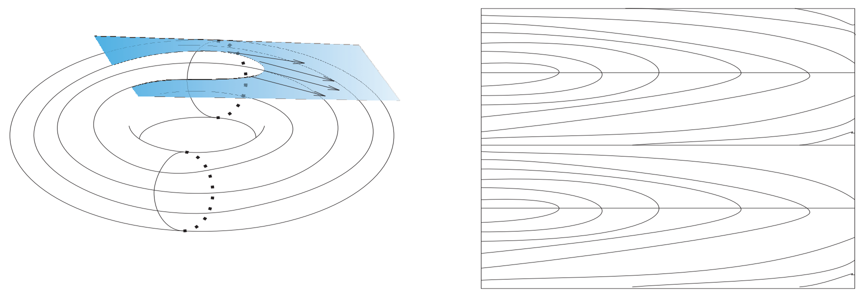

Figura 6.3: Two Foliations's of Reeb on the torus ∂U_l

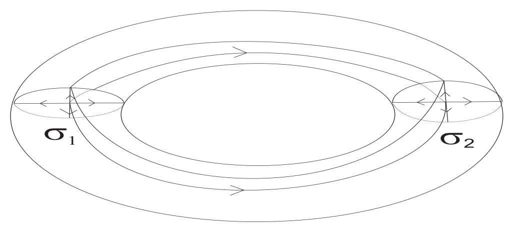

Figura 6.4: Solid Torus $M^2 = D^2 \times S^1$.

where σ_1, σ_2 are singularities, A is an attractor, R is a repeller both non trivial hyperbolics. The set $A \cup R \cup W^u(\sigma_2)$ works the Theorem. ■

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